A boundary element method for the hydrodynamic analysis of floating bodies in variable bathymetry regions

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Abstract

In this work, a hybrid technique is presented for the hydrodynamic analysis of floating bodies in variable bathymetry regions. Our method is based on the coupled-mode theory for the propagation of water waves in general bottom topography, developed by Athanassoulis and Belibassakis [Athanassoulis GA, Belibassakis KA. A consistent coupled-mode theory for the propagation of small-amplitude water waves over variable bathymetry regions. J Fluid Mech 1999;389:275–301.] and extended to 3D by Belibassakis et al. [Belibassakis KA, Athanassoulis GA, Gerostathiis TP. A coupled-mode model for the refraction-diffraction of linear waves over steep three-dimensional bathymetry. Appl Ocean Res 2001;23:319–336.], which is free of any mild-slope assumption, in conjunction with a boundary integral representation of the near field in the vicinity of the floating body. Both 2D and 3D problems have been considered. In all cases the near field is represented by boundary integral representation involving simple (Rankine) sources. In the 2D case, the far-field is modelled by complete (normal-mode) series expansions derived by separation of variables in the constant-depth half-strips. In the 3D case, the far-field is modelled by an integral representation involving the appropriate Green’s function for harmonic water waves over a bottom with different depths at infinity, developed by Belibassakis and Athanassoulis [Belibassakis KA, Athanassoulis GA. Three-dimensional Green’s function for harmonic water waves over a bottom with different depths at infinity. J Fluid Mech 2004;510:267–302.]. The numerical solution is obtained by means of a low-order panel method materialising the hybrid technique. Numerical results are presented concerning floating bodies of simple geometry, lying over sloping and undulating seabeds. With the aid of systematic comparisons with benchmark solutions the convergence and accuracy of the present method in 3D has been studied, and the effects of bottom slope and curvature on the hydrodynamic characteristics (hydrodynamic coefficients and responses) of the floating bodies are illustrated and discussed.

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1. Introduction

The interaction of free-surface gravity waves with floating bodies, in water of intermediate depth and in variable bathymetry regions, is a mathematically interesting problem finding important applications. A specific example is the design and evaluation of performance of special-type ships and structures operating in nearshore and coastal waters; see, e.g., Mei [1], Sawaragi [2]. Also, pontoon-type floating bodies of relatively small dimensions find applications as coastal protection devices (floating breakwaters) and are also frequently used as small boat marinas; see, e.g., Williams [3], Drimer et al. [4], Williams et al. [5], Drobyshhevski [6,7]. In all these cases, the estimation of wave-induced loads and motions of the floating structure can be based on the solution of classical wave–body–seabed hydrodynamic interaction problems; see, e.g., Wehausen [8,9].

Theoretical aspects of the problem of small-amplitude water waves propagating in finite water depth and their interaction with floating and/or submerged bodies, mainly existence and uniqueness results, have been presented by various authors; see, e.g., Mei [1], Kuznetsov et al. [10], and Evans and Kuznetsov [11] as concerns the existence of trapped modes in a channel with obstructions. In the case of diffraction of water waves by vertical cylindrical structures (extending throughout the water column) in constant depth, the problem can be reduced to the Helmholtz equation on the horizontal plane and it can be very conveniently treated by applying BEM, as presented by Au and Brebbia [12]. This technique has been further extended by Zhu [13] (see also Ref. [14]) using the dual reciprocity boundary element method to study combined refraction and diffraction of water waves around solid offshore structures over a seabed with variable depth, under the mild-slope assumption concerning the bottom surface.
Moreover, intermediate-depth and shallow-water conditions are frequently encountered in marine applications. When barges, offshore structures or floating docks are moored or towed in nearshore and shallow areas, accurate prediction of motions is required in order to design the mooring system and to insure that under-keel clearance is sufficient for the structure to avoid grounding. In most applications, the water depth is assumed to be constant, which is practically valid in the case when the depth variation is small or the horizontal dimension(s) of the floating body are small in comparison to the bottom variation length. However, in applications involving the utilisation of floating bodies in coastal waters, the variations of bathymetry may have a significant effect on the hydrodynamic behaviour of ships and structures. In this connection, under the assumption of slowly varying bathymetry, mild-slope models have been developed for the analysis of wave-induced ship (or floating body) motion, see, e.g. Takagi et al. [15]. This approach has been further extended by Ohyama and Tsuchida [16] to treat the problem in environments characterised by steeper bathymetric variations, as, e.g., in the entrance of ports and harbours.

In the present work, a hybrid technique is developed and applied to the hydrodynamic analysis of floating bodies in variable bathymetry regions. Following the hybrid formulation by Yeung et al. [17], the present method utilises the simplicity of the Rankine sources, in conjunction with appropriate representations of the wave field in the exterior infinite domain(s), similarly as presented by Nestegard and Sclavounos [18] for 2D radiation problems in deep water and by Drimer and Agnon [19] in the case of finite water depth, as well as by Yuen and Chau [20] and Matsui et al. [21] for 3D problems. It must be noted, however, that the last two works are based on complete cylindrical expansions in the outer domain and thus, they are restricted in cases concerning three-dimensional floating scatterers and/or inclusions in constant depth. Under the assumption of a slowly varying bathymetry, BEM techniques, in conjunction with the mild-slope equation, have been also recently applied to study combined refraction and diffraction of water waves around floating structures located in the coastal zone; see, e.g., Ref. [22].

To overcome the mild-slope assumption, the present method is based on the coupled-mode theory for the propagation of water waves in variable bathymetry regions, developed by Athanassoulis and Belibassakis [23] and extended to 3D by Belibassakis et al. [24], in conjunction with a boundary integral representation of the near field in the vicinity of the floating body and an appropriate representation of the far-field flow dynamics in regions with non-constant depth at infinity. Both 2D and 3D problems have been considered. In all cases, the near field is represented by boundary integral representation involving simple (Rankine) sources. The numerical solution is obtained by means of a low-order panel method materialising the hybrid technique.

In order to better illustrate the present approach, first the simpler 2D problem is considered. In this case, the far-field is modelled by complete (normal-mode) series expansions derived by separation of variables in the two constant-depth half-strips separating the variable bathymetry region from the regions of wave incidence and wave transmission (see Fig. 1). Numerical results are presented concerning floating bodies of simple geometry lying over sloping and undulating beds. With the aid of systematic comparisons, the effects of bottom slope and curvature on the hydrodynamic characteristics (hydrodynamic coefficients and responses) are presented and discussed. Then, the 3D problem is considered, which is more complicated by the fact that the physical properties of the medium (the index of refraction and the propagation speed) are different at infinity as approached from different directions. In this case, the far-field wave pattern is not known a priori, and a standard radiation condition (e.g., Sommerfeld condition) cannot be applied. This difficulty is overpassed by means of a suitable boundary integral representation, involving the 3D Green’s function for harmonic water waves over an uneven bottom topography, recently developed by Belibassakis and Athanassoulis [25]. The latter permits us to consistently formulate and solve the wave–body–seabed interaction problems, in variable bathymetry regions with different depths at infinity, by a suitable extension of the hybrid BEM to 3D.

2. The 2D problem

We first consider the 2D problem concerning the hydrodynamic behaviour of a floating body $D_B$ of arbitrary cross ($yz$) section in the coastal-marine environment illustrated in Fig. 1. This environment consists of a water layer bounded above by the free surface and below by a rigid bottom. It is assumed that the bottom surface exhibits an arbitrary one-dimensional variation, i.e., the bathymetry is characterised by parallel, straight bottom contours lying between two regions of constant but different depth, $h = h_1$ (the deeper water region or region of incidence) and $h = h_2$ (the shallower water region or region of transmission). A Cartesian coordinate system $\{x, y, z\}$ is introduced, with its origin at some point on the mean water level (in the variable bathymetry region), coinciding with the centre of flotation of the floating body, and the $z$-axis pointing upwards. The liquid domain $D$ is decomposed in three parts $D^j_{\alpha}$, $j = 1, 2, 3$ (see Fig. 1), defined as follows: $D^{(1)}$ is the subdomain characterised by $y < a$ where the depth is constant and equal to $h_1$, and $D^{(3)}$ is the subdomain characterised by $y > b$ where the depth is constant and equal to $h_3$. Also, $D^{(2)}$ is the variable bathymetry subdomain ($a < y < b$) containing the floating and/or immersed body. The depth function characterising this environment is assumed to be smooth and has the following form:

$$h(y) = \begin{cases} h_1, & y \leq a, \\ h_2(y), & a < y < b, \\ h_3, & y \geq b. \end{cases} \quad (2.1)$$

The liquid is assumed homogeneous, inviscid and incompressible. The wave field in the region $D$ is excited by a harmonic incident wave, with direction of propagation normal to the depth-contours (along the $y$-axis). Without loss of generality, a left-incident wave has been assumed; see Fig. 1. Thus, the fluid motion is described by the 2D wave potential $\phi(y, z; t)$. The latter, under the assumptions that the free-surface elevation and the wave velocities are small, satisfies the linearised water wave equations; see, e.g., Wehausen and Laitone [26] or Mei [1]. Then, the wave field is time harmonic

$$\phi(y, z; t) = \text{Re}\left\{ -\frac{igH}{2\rho} \psi(y, z; \mu) \exp(-i\omega t) \right\}, \quad \text{(2.2a)}$$

The geometrical configuration concerning the 2D problem is shown in Fig. 1.
where \( H \) is the incident wave height, \( g \) is the acceleration due to gravity, \( \mu = \omega^2/g \) is the frequency parameter, and \( i = \sqrt{-1} \). The free-surface elevation is obtained in terms of the wave potential on the free surface as follows:

\[
\eta(y, t) = -\frac{1}{g} \frac{\partial \phi(y, z = 0)}{\partial t}.
\] (2.2b)

The function \( \phi = \phi(y, z; \mu) \), appearing in Eq. (2.2), is the normalised potential in the frequency domain, simply written as \( \phi(y, z) \). Using standard floating-body hydrodynamic theory [8,27], the potential \( \phi(y, z) \) can be decomposed as follows:

\[
\phi(y, z) = \phi_0(y, z) + \phi(y, z) + \frac{\mu H}{\pi} \sum_{k} c_k \phi_k(y, z),
\] (2.3)

where \( \phi_0(y, z) \) denotes the propagation wave potential, corresponding to the solution of the water-wave problem in the variable bathymetry region in the absence of the scatterer \( D_B \). Moreover, \( \phi(y, z) \) is the diffraction wave potential in the non-uniform strip \( D \) due to the presence of the fixed body \( D_B \), and \( \phi_k(y, z), k = 2, 3, 4 \), stand for the radiation potentials associated with the 3 degrees of freedom concerning the motion of the floating body, i.e. sway (\( k = 2 \)), heave (\( k = 3 \)) and roll (\( k = 4 \)). Finally, \( c_k, k = 2, 3, 4 \), denote the complex amplitudes of the corresponding motions of the floating body.

In the sequel, the wave potential \( \phi(y, z) \) associated with the propagation/diffraction of water waves in the variable bathymetry region without the scatterer (floating body) will be also called the \textit{generalised incident wave} field. This potential is considered to be known, as, e.g., obtained through the application of the consistent coupled-mode theory developed by Athanassoulis and Belibassakis [23] which is briefly described in Section 3 of this work. The rest of sub-problems on \( \phi(y, z), k = 2, 3, 4 \), in the variable bathymetry region, can be formulated as \textit{radiation-type} problems in the bounded subdomain \( D^{(3)} \), with the aid of the following general representations of the wave potential \( \phi(y, z) \) in the semi-infinite strips \( D^{(1)} \) and \( D^{(3)} \) which are obtained by separation of variables (see, e.g., Ref. [28]):

\[
\phi_k(y, z) = C^{(1)}_k \exp(-i k^1 y) Z^{(1)}_0(z) + \sum_{n=1}^{\infty} C^{(2)}_n \exp(i k_n^1 y) Z^{(1)}_n(z),
\] (2.4)

\[
\phi_0(y, z) = C^{(2)}_0 \exp(i k^1_0 y) Z^{(1)}_0(z) + \sum_{n=1}^{\infty} C^{(3)}_n \exp(i k_n^1 y) Z^{(1)}_n(z).
\] (2.5)

The first term (\( n = 0 \)) in the series (2.4) and (2.5), respectively, are the propagating modes, while the remaining ones (\( n = 1, 2, . . . \)) are the evanescent modes. In the expansions (2.4) and (2.5) the sets of numbers \( \{ik^2_0, k_n^2\}, n = 1, 2, . . . , i = 1, 3 \), and the sets of vertical functions \( \{Z^{(2)}_n(y), n = 0, 1, 2, . . . , i = 1, 3\} \), \( i = 1, 3 \), are, the eigenvalues and the corresponding eigenfunctions of regular Sturm–Liouville problems obtained by separation of variables in the constant-depth half-strips \( D^{(2)} \). The eigenvalues \( \{k_0^2, k_n^2\} \) are given as the roots of the dispersion relations

\[
\mu_i h_i = -k_0^2 h_i \tan(k_0^2 h_i); \quad n = 0, 1, 2, . . . , \quad i = 1, 3,
\] (2.6a,b)

and the eigenfunctions \( \{Z^{(2)}_n(y), n = 0, 1, 2, . . . , i = 1, 3\} \) are given by

\[
Z^{(2)}_0(z) = \frac{\cosh(k^2_0 (z + h_i))}{\cosh(k^2_0 h_i)}, \quad Z^{(2)}_n(z) = \frac{\cosh(k^2_n (z + h_i))}{\cosh(k^2_n h_i)}.
\] (2.7)

The completeness of the expansions (2.4) and (2.5) follows by the standard theory of regular eigenvalue problems; see, e.g., Coddington and Levinson [29]. On the basis of these representations, the hydrodynamic problems concerning the diffraction and all radiation potentials \( \phi_k(y, z), k = 0, 2, 3, 4 \) can be formulated as \textit{radiation-type} problems, satisfying the following systems of equations, boundary and matching conditions:

\[
\frac{\partial^2 \phi_k}{\partial y^2} + \frac{\partial^2 \phi_k}{\partial z^2} = 0, \quad (y, z) \in D^{(2)},
\] (2.8a)

\[
\frac{\partial \phi_k}{\partial n} - i \mu \phi_k = 0, \quad (y, z) \in \partial D^{(2)},
\] (2.8b)

\[
\frac{\partial \phi_k}{\partial n} = 0, \quad (y, z) \in \partial D^{(2)}_n,
\] (2.8c)

\[
\frac{\partial \phi_k}{\partial n} = g_k, \quad (y, z) \in \partial D_B,
\] (2.8d)

\[
\frac{\partial \phi_k}{\partial n} - T_{12}[\phi_k] = 0, \quad (y, z) \in \partial D^{(1)}_{a, b},
\] (2.8e)

\[
\frac{\partial \phi_k}{\partial n} - T_{23}[\phi_k] = 0, \quad (y, z) \in \partial D^{(3)}_n,
\] (2.8f)

where \( \partial D^{(2)}_a \) and \( \partial D^{(2)}_b \) denote the free-surface and bottom parts of the domain \( D^{(2)} \), \( \partial D_B \) denotes the (solid) boundary of the floating body, and \( \partial D^{(1)}_a, \partial D^{(3)}_n \) are the vertical interfaces located at \( y = a \) and \( y = b \) (respectively), separating the three-subdomains \( D^{(n)}_a, i = 1, 2, 3 \). The latter are also shown by using vertical dashed lines in Fig. 1. Moreover, in Eqs. (2.8a) \( n = (0, n_1, n_2) \) denotes the unit normal vector to the boundary \( \partial D^{(2)}_a \), directed to its exterior. The operators \( T_{12}[\phi_k] \) and \( T_{23}[\phi_k] \) are appropriate \textit{Dirichlet-to-Neumann} (DtN) maps, see, e.g., Givoli [30], ensuring the complete matching of the wave fields \( \phi_k(y, z), k = 0, 2, 3, 4 \), on the vertical interfaces \( \partial D^{(1)}_a, \partial D^{(3)}_n \), respectively. These operators are constructed from Eqs. (2.4) and (2.5), respectively, exploiting the completeness properties of the vertical bases \( \{Z^{(n)}_n(y), n = 0, 1, 2, . . . , i = 1, 2, \ldots, i, i = 1, 2, \ldots, i, i = 1, 2, \ldots, i\} \), \( i = 1, 2, \ldots, i, i = 1, 2, \ldots, i \), and are given as follows:

\[
T_{12}[\phi_k] = ik^{12}_0 Z^{(1)}_0(z) \int_{z = -h}^{z = b} \phi_k(y, a, z) Z^{(1)}_n(z) \, dz
\]

\[
- \sum_{n=1}^{\infty} k^{12}_n Z^{(1)}_n(z) \int_{z = -h}^{z = b} \phi_k(y, a, z) Z^{(1)}_n(z) \, dz,
\] (2.9)

\[
T_{23}[\phi_k] = ik^{13}_0 Z^{(2)}_0(z) \int_{z = -h}^{z = b} \phi_k(y, b, z) \sum_{n=1}^{\infty} k^{13}_n Z^{(2)}_n(z) \, dz
\]

\[
\times \int_{z = -h}^{z = b} \phi_k(y, b, z) Z^{(3)}_n(z) \, dz Z^{(3)}_n(z) \, dz,
\] (2.10)

where \( Z^{(n)}_n(z) = Z^{(n)}_n(z)/|Z^{(n)}_n(z)| \) are the normalised eigenfunctions, and \( |Z^{(n)}_n(z)| = (\int_{z = -h}^{z = b} (Z^{(n)}_n(z))^2 \, dz)^{1/2} \). The L₂-norm of each vertical eigenfunction.

Finally, the boundary data \( g_k, k = 0, 2, 3, 4 \), appearing in the right-hand side of Eq. (2.8d), constitute the forcing of the systems (2.8) for each \( k \). These data are defined by the normal derivative of the generalised incident field, \( g_k = -\partial \phi_0/\partial n \), and by the components of the generalised normal vector on the body boundary: \( g_n = n_1, g_3 = n_2, g_4 = y_n \), all calculated on the wetted surface \( \partial D_B \) of the floating body.

3. Calculation of the generalised incident wavefield

The wave potential \( \phi_k(y, z) \) associated with the propagation of water waves in the variable bathymetry region, without the presence of the scatterer (floating body), can be conveniently calculated by means of the consistent coupled-mode model developed by Athanassoulis and Belibassakis [23]. This model is
based on the following enhanced local-mode representation:
\[
\phi_\text{P}(y, z) = \phi_{-1}(y)Z_{-1}(z; y) + \phi_0(y)Z_0(z; y) + \sum_{n=1}^{\infty} \phi_n(y)Z_n(z; y). 
\]
(3.1)

In the above expansion, the term \( \phi_0(y)Z_0(z; y) \) denotes the propagating mode of the generalised incident field. The remaining terms \( \phi_n(y)Z_n(z; y), n = 1, 2, \ldots \), are the corresponding evanescent modes, and the additional term \( \phi_{-1}(y)Z_{-1}(z; y) \) is a correction term, called the sloping-bottom mode, which properly accounts for the satisfaction of the Neumann bottom boundary condition on the non-horizontal parts of the bottom. The function \( Z_0(z; y) \) represents the vertical structure of the nth mode. The function \( \phi_n(y) \) describes the horizontal pattern of the nth mode and is called the complex amplitude of the nth function. The functions \( Z_n(z; y) \), \( n = 0, 1, 2, \ldots \), appearing in Eq. (3.1) are obtained as the eigenfunctions of local vertical Sturm–Liouville problems (formulated for each \( y \) in a \( a \leq y \leq b \)), and are given by

\[
Z_0(z; y) = \frac{\cosh[k_0(y)(z + h(y))]}{\cosh(k_0h(y))}, \quad Z_n(z; y) = \frac{\cosh[k_n(y)(z + h(y))]}{\cosh(k_nh(y))}, \quad n = 1, 2, \ldots. 
\]
(3.2a)

Moreover, the eigenvalues \( ik_k(y), k_n(y) \) are obtained as the roots of the local dispersion relation

\[
\mu h(y) = -k(y)h(y)\tan(k(y)h(y)), \quad a \leq y \leq b. 
\]
(3.2b)

A specific convenient form of the function \( Z_{-1}(z; y) \) associated with the sloping-bottom mode is given by

\[
Z_{-1}(z; y) = h(y) \left( \frac{z}{h(y)} \right)^3 + \frac{z}{h(y)} \right)^2
\]
(3.2c)

and all numerical results presented in this work are based on this choice for \( Z_{-1}(z; y) \). However, other choices are also possible; see the discussion by Athanassoulis and Belibassakis [23]. By following the procedure described in the latter work, the coupled-mode system of horizontal equations for the amplitudes of the generalised incident wave field is finally obtained,

\[
\sum_{n=1}^{\infty} A_{mn}(y)\phi_n(y) + B_{mn}(y)\phi_0(y) + C_{mn}(y)\phi_n(y) = 0, \quad m = -1, 0, 1, \ldots, 
\]
(3.3)

in \( a < y < b \), where a prime denotes differentiation with respect to \( y \). The coefficients \( A_{mn}, B_{mn}, C_{mn} \) of the coupled-mode system (3.3) are dependent on \( y \) through \( h(y) \) and can be found in Table 1 of Athanassoulis and Belibassakis [23]. The coupled-mode system (3.3) is supplemented by the following boundary conditions

\[
\phi_{-1}(a) = \phi'_{-1}(a) = 0, \quad \phi_{-1}(b) = \phi'_{-1}(b) = 0, 
\]
(3.4a)

\[
\phi_0'(a) + ik_0^{(1)}\phi_0(a) = 2ik_0^{(1)}\exp(ik_0^{(1)}a), 
\phi_0'(b) - ik_0^{(3)}\phi_0(b) = 0, \quad n = 1, 2, 3, \ldots. 
\]
(3.4b)

\[
\phi_0'(a) + ik_0^{(1)}\phi_0(a) = 0, \quad n = 1, 2, 3, \ldots. 
\]
(3.4c)

The above conditions ensure the complete matching at \( y = a \) and \( y = b \), respectively, of the wave potential represented by Eq. (3.1) in the variable bathymetry with the representations (2.4) in the two semi-infinite strips. The solution of the coupled-mode system (3.3) provides us with the modal amplitudes \( \phi_n(y), n = -1, 0, 1, 2, \ldots \), which substituted in the local-mode expansion (3.1) enable the calculation of the generalised incident wave potential \( \phi_\text{P}(y, z) \) at every point in the domain. Also, on the basis of the solution of the coupled-mode system, the reflection and transmission coefficients \( (A_R, A_T) \) of the waves in the environment (without the presence of the floating body) can be calculated from the amplitude of the propagating mode \( \phi_0(y) \) at the ends of the variable bathymetry region \( (y = a \text{ and } y = b) \) as follows:

\[
A_R = |\phi_0(a) - \exp(ik_0^{(1)}a)| \exp(ik_0^{(1)}a), 
A_T = |\phi_0(b) - \exp(-ik_0^{(3)}b)|. 
\]
(3.5a)

An important feature of the calculation of the generalised incident field by means of the enhanced representation (3.1) is that the modal amplitudes \( |\phi_n(y)| \) exhibit an improved rate of decay of the order \( O(n^{-4}) \). Thus, only a few number of modes \( (5 \text{ or } 6) \) suffice to obtain a convergent solution to \( \phi_\text{P}(y, z) \), for bottom slopes of the order of 1:1, or even higher. More details about the above coupled-mode technique can be found in Athanassoulis and Belibassakis [23], and as concerns its application to realistic 3D seabed topographies in Belibassakis et al. [24] and in Gerostathi et al. [31].

4. The diffraction and radiation problems

The corresponding problems on the diffraction and radiation potentials \( \phi_\text{P}(y, z) \), \( k = 0, 2, 3, 4 \), Eqs. (2.8), are treated by means of boundary integral equation formulations, based on the single layer potential [see, e.g., Ref. [9]]. Accordingly, the following integral representations are introduced for \( \phi_\text{P}(y, z) \) of \( k = 0, 2, 3, 4 \), in the variable bathymetry subdomain \( D^{(2)} \):

\[
\phi_\text{R}(r) = \int_{\partial D^{(2)}} \sigma_k(r')G(r', r) \, d\Gamma(r'), 
\]
(4.1)

where \( G(r', r) = \ln |r'| - r'/2\pi \) is the Green’s function of the Laplace equation in 2D free space, and \( d\Gamma(r') \) denotes the differential element along the boundary \( \partial D^{(2)} \); see, e.g., Kress [32], Katz-Plotkin [33]. Based on the properties of the single-layer distributions, the corresponding normal derivatives of the potentials \( \phi_\text{P}(y, z) \), \( k = 0, 2, 3, 4 \), on the boundary \( \partial D^{(2)} \) are given by (see, e.g., Ref. [32]):

\[
\sigma_k(r) = \frac{\partial \phi_\text{R}(r)}{\partial n}, \quad r \in \partial D^{(2)}, 
\]
(4.2)

Using Eqs. (4.1) and (4.2) in Eqs. (2.8) we obtain, for each one of the potentials \( \phi_\text{P}(y, z) \), \( k = 0, 2, 3, 4 \), a system of boundary integral equations with support on various parts of \( \partial D^{(2)} \) for the determination of the corresponding unknown source distribution \( \sigma_k(r), r \in \partial D^{(2)} \). The final system (for \( k = 0, 2, 3, 4 \) reads as follows:

\[
-\sigma_k(r) + \int_{\partial D^{(2)}} \sigma_k(r')G(r', r) \, d\Gamma(r') = 0, \quad r \in \partial D^{(2)}, 
\]
(4.3a)

\[
-\sigma_k(r) + \int_{\partial D^{(2)}} \sigma_k(r')G(r', r) \, d\Gamma(r') = 0, \quad r \in \partial D^{(2)}, 
\]
(4.3b)

\[
-\sigma_k(r) + \int_{\partial D^{(2)}} \sigma_k(r')G(r', r) \, d\Gamma(r') = g_k(r), \quad r \in \partial D_k, 
\]
(4.3c)

\[
-\sigma_k(r) + \int_{\partial D^{(2)}} \sigma_k(r')G(r', r) \, d\Gamma(r') = -T_{12} \int_{\partial D^{(2)}} \sigma_k(r')G(r', r) \, d\Gamma(r'), \quad r \in \partial D^{(2)}, 
\]
(4.3d)
The normalisation used for the added-mass coefficients plotted in these figures is 

\( \frac{\rho}{gH} \), where \( \rho \) is the fluid density, \( g \) is the acceleration due to gravity, \( H \) is the incident wave height, and \( d \) is the displacement of the floating body.

To illustrate the effect of bottom slope on the hydrodynamic characteristics of floating bodies in variable bathymetry regions, we consider the case of a floating pontoon, i.e., a floating body of rectangular cross-section, with non-dimensional breadth \( B/h = 1.5 \) and draft \( d/h = 0.5 \), where \( h \) denotes the (mean) depth, floating in a variable bathymetry region; see Fig. 2(a). The centre of gravity has been selected to coincide with the centre of flotation \( (KG = d) \) and thus, the (non-dimensional) metacentric height of this structure is \( GM/h = 0.625 \). Numerical results are presented in Figs. 2 and 3 concerning the hydrodynamic behaviour of this floating structure in constant depth and over two linear shoals characterised by (constant) bottom slopes 12.5% and 25%, respectively; see Fig. 2(a). In particular, in Fig. 2 we present the effect of bottom slope on the hydrodynamic forces, on the added masses and on the damping coefficients, respectively. More specifically, in Fig. 2(b) the normalised hydrodynamic forces \( F_k \), \( k = 2, 3, 4 \) are presented vs. the non-dimensional wavelength \( \lambda/h \), where \( \lambda = 2\pi/k_0 \) is the wavelength corresponding to the mean water depth \( h \), as obtained through the application of the linearised dispersion relation (Eq. (2.6), for \( n = 0 \)). The normalisation used for the hydrodynamic forces is 

\[ \tilde{F}_k = \frac{F_k}{\rho g h^3}, \quad k = 2, 3, 4. \]  

where \( H \) is the incident wave height. Results concerning the three bottom profiles are shown by using solid lines (flat bottom), dashed-dotted lines (sloping seabed 12.5%), and dashed lines (sloping seabed 25%). Moreover, in Fig. 2(c)–(f) corresponding results concerning the hydrodynamic coefficients are presented. The normalisation used for the added-mass coefficients plotted in these figures is 

\[ \tilde{a}_{kl} = \frac{a_{kl}}{\rho h^2}, \quad k, l = 2, 3, 4. \]  

and for the damping coefficients:

\[ \tilde{b}_{kl} = \frac{b_{kl}}{\rho h^2 \sqrt{g/h}}, \quad k, l = 2, 3, 4. \]  

where \( \tilde{a}_{kl} \) and \( \tilde{b}_{kl} \) are the normalised added-mass and damping coefficients, respectively.

In the above equations \( h \) denotes the (mean) water depth, which is the same with the depth below the centreline of the floating body (see Fig. 2(a)). Furthermore, in Fig. 3(a), the effect of bottom slope is illustrated concerning the transmission coefficient \( (A_0) \) of the rectangular pontoon, considered to be either freely floating or fixed. In the second case no motions are permitted \( (\zeta = 0, k = 2, 3, 4) \). It is stressed here that the transmission coefficient is a quantity of interest for this kind of floating bodies, especially in connection with their utilisation as floating breakwaters or boat marinas. In the rest subplots of Fig. 3, the effect of bottom slope on
the motions of the freely floating body is examined. In this case, the response amplitude operators (RAOs) of the floating pontoon in sway, heave and roll motion are shown, as calculated by the present method, over the flat and the two sloping seabeds. All the above results are parametrically plotted vs. the non-dimensional wavelength $\lambda/h$, where $h$ denotes the average water depth.

**Fig. 2.** (a) The floating body of rectangular cross-section ($B/h = 1.5$, $d/h = 0.5$) and the three sloping-bottom profiles considered. (b) Hydrodynamic forces $F_k$, $k = 2, 3, 4$, and (c–f) various hydrodynamic coefficients vs. the non-dimensional wavelength $\lambda/h$. All quantities are plotted vs. the non-dimensional wavelength $\lambda/h$, where $h$ denotes the average water depth.

**Fig. 3.** (a) Transmission coefficient of fixed and freely floating body of rectangular cross-section $B/h = 1.5$, $d/h = 0.5$, in flat seabed (solid line), and sloping seabeds 12.5% (dashed-dotted line) and 25% (dashed line). (b, c) Floating body RAOs in sway and heave motion ($2|\gamma_k|/H, k = 2, 3$), respectively. (d) Floating body RAO in roll motion ($2|\gamma_4|/k_0H$). All quantities are plotted vs. the non-dimensional wavelength $\lambda/h$, where $h$ denotes the average water depth.
motions sway and heave, respectively,
\[ \frac{z_k}{a_k} = \frac{2|\zeta_k|}{H}, \quad k = 2, 3, \]  
(5.7a)
are shown vs. the non-dimensional wave length \( \lambda/h \), while the corresponding RAO concerning the angular roll motion
\[ \frac{z_4}{a_4} = \frac{2|\zeta_4|}{H}, \]  
(5.7b)
is shown in Fig. 3(d). From the results presented in Figs. 2 and 3, we can clearly observe that the effects of bottom slope on the hydrodynamic characteristics of the floating bodies can become significant, especially in intermediate and shallow water conditions \( (\lambda/h > 7) \). This result could be important concerning the design and performance of ships and offshore structures operating in nearshore and coastal waters.

### 5.2. The effect of bottom undulations

Similar results concerning the same floating pontoon are presented in Figs. 4 and 5, but in this case for undulating seabeds, corresponding to a sinusoidal bottom topography of the form:
\[ h(y) = h \left( 1 - a_b \cos \left( \frac{2\pi y}{\lambda_b H} \right) \exp \left( \gamma_b y^2 \right) \right), \]  
(5.8)
where \( a_b = 0.2 \) and \( 0.3 \) is the (non-dimensional) amplitude of bottom corrugations, \( \lambda_b = 1.4 \) is the corresponding wavelength, and \( \gamma_b \) is a coefficient expressing the decaying of bottom corrugations (moving away from the region of the floating body).

From Figs. 4 and 5, we see that the effects of the bottom undulations can also become quite important, especially in intermediate and shallow water conditions \( (\lambda/h > 5-7) \). It is observed from Fig. 4 that the bottom corrugations affect more significantly the hydrodynamic moment \( F_4 \), the added-mass coefficient associated with heave motion \( (a_{33}) \) and the corresponding coefficients of coupled sway-roll motion \( (a_{24} \text{ and } b_{24}) \). Also, from Fig. 5 we see that the magnitude of bottom corrugations has a significant effect on the transmission coefficients and the responses of the floating body with respect to all degrees of freedom. On the other hand, bottom corrugations do not significantly change the frequency of resonance, which in the present example remains for all cases considered, \( \omega \sqrt{h/g} = 0.8 \).

### 6. The 3D problem

In this section, we shall extend the present theory by developing a hybrid scheme to treat the problem of hydrodynamic analysis of a 3D body, floating over an uneven seabed which is characterised by parallel bottom contours. The task is accomplished by means of the 3D bottom-dependent Green's function \( G(\mathbf{r}, \mathbf{r}_0) \) developed recently by Belibassakis and Athanassoulis [25]. This singularity satisfies the no-entrance boundary condition on the bottom profile and exhibits the appropriate behaviour at infinity. The calculation of \( G(\mathbf{r}, \mathbf{r}_0) \) is quite cumbersome task, which does not permit its direct application to formulate and solve a single boundary integral equation with support only on the wetted surface of the floating body, similarly as in the case of 3D bodies in a constant-depth strip; see, e.g., Lee and Newman [36].

In the context of hybrid methods (also discussed in Ref. [1], Chap. 7.7), different representations of the solution to the diffraction and radiation problems associated with the floating body hydrodynamics are used in the near field \( D_N \), which encloses the body and the variable bathymetry region, and in the far-field \( D_D \); see Fig. 6. Here, the two subdomains are separated by means of the cylinder of radius \( \rho_C \), decomposing the whole liquid.

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**Fig. 4.** (a) Floating body of rectangular cross-section \((B/h = 1.5, \ d/h = 0.5)\) above flat bottom and undulating bottom profiles. (b) Hydrodynamic forces \( \tilde{F}_k, k = 2, 3, 4 \), and (c–f) normalised hydrodynamic coefficients. All quantities are plotted vs. the non-dimensional wavelength \( \lambda/h \), where \( h \) denotes the average water depth.
domain \( D \) into the interior (near field) \( D_N \) and the exterior (far-field) part \( D_D \). Extending our previous 2D analysis, one possible hybrid scheme can be formulated by using simple (Rankine) sources for the diffraction/radiation wave potentials in \( D_N \):

\[
\phi(r) = \int_{\partial D_B} \sigma_N(r_0) G_N(r_0, r) \, dS(r_0), \quad r \in D_N, \tag{6.1}
\]

where \( G_N = 1/4\pi |r-r_0| \), denotes the 3D simple source, in conjunction with a representation of the wave field in \( D_D \) by means of a distribution of bottom-dependent sources \( G(r_0) \) on the wetted surface of the floating body:

\[
\phi(r) = \int_{\partial D_B} \sigma(r_0) G(r_0, r) \, dS(r_0), \quad r \in D_D. \tag{6.2}
\]

Subdividing the wetted surface of the body, \( \partial D_B \), in a union of sufficiently small, disjoint boundary elements \( \{E_j, j = 1, J\} \), and taking the distance between the body and the cylindrical surface \( \rho = \rho_C \) to be large enough, the following approximation for the wave potential in the exterior domain is easily obtained:

\[
\phi(r) = \int_{E_j} \sigma(r_0) G(r_0, r) \, dS(r_0) \approx \sum_{j=1}^J \sigma(r_{0j}) G(r_{0j}, r) S_j, \tag{6.3}
\]

where \( r \in D_N \). In the above equation, \( r_{0j} \) denotes the centroid of the surface panel \( E_j \) and \( S_j \) denotes its area. Then, the diffraction–refraction–radiation problems associated with the wave–body–seabed interaction can be solved by means of the following system of equations (defining the hybrid scheme):

\[
-\frac{\sigma_N(r)}{2} + \int_{\partial D_B} \sigma_N(r_0) \frac{\partial G_N(r_0, r)}{\partial n} \, dS(r_0) = \begin{cases} g(r), & r \in \partial D_B, \\ 0, & r \in \partial D_N, \end{cases} \tag{6.4}
\]

\[
-\frac{\sigma_N(r)}{2} + \int_{\partial D_B} \sigma_N(r_0) \left( \frac{\partial G_N(r_0, r)}{\partial n} - \mu G_N(r_0, r) \right) \, dS(r_0) = 0, \quad r \in \partial D_D, \tag{6.5}
\]

where \( \partial D_B, \partial D_N \) denote the free-surface and bottom boundary parts of \( D_N \), respectively, and \( \mathbf{n} \) is the unit normal vector directed to the exterior of \( D_B \). The Neumann data on the solid surface of the floating body \( r \in \partial D_B \) for the diffraction problem are

\[
g_0(r) = -\frac{\partial \phi}{\partial n} \quad k = 0, \tag{6.6a}
\]

and as concerns the radiation problems

\[
g_k(r) = n_k, \quad k = 1, 2, 3 \quad (\text{surge, sway, heave motions}), \tag{6.6b}
\]

\[
g_{3+k}(r) = (r \times n)_k, \quad k = 1, 2, 3 \quad (\text{roll, pitch, yaw motions}). \tag{6.6c}
\]

Again, the function \( g_0 \) is obtained from the solution \( (\phi_P) \) of the wave propagation–refraction problem over the variable

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**Fig. 5.** (a) Transmission coefficient of fixed and freely floating body of rectangular cross-section \( B/h = 1.5, d/h = 0.5 \), in flat seabed \( (x_d = 0, \text{solid line}) \), and undulating seabeds with \( x_d = 0.2 \) (dashed-dotted line) and \( x_d = 0.3 \) (dashed line). (b, c) Floating body RAOs in sway and heave motion \( (2|z_n|/H, k = 2, 3 \), respectively. (d) Floating body RAO in roll motion \( (2|z_n|/k_0 H) \). All quantities are plotted vs. the non-dimensional wavelength \( \lambda/h \), where \( h \) denotes the average water depth.

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**Fig. 6.** Definition of the hybrid scheme for the 3D problem.
bathymetry region, without the presence of the body (in the case of the diffraction problem), as described in Section 3. The functions \( g_k, k = 1, 2, \ldots, 6 \) associated with the radiation problems are defined by the components of the generalised normal vector on the surface of the floating body \( \partial D_0 \). The formulation of the hybrid scheme is completed by the matching conditions, requiring the continuity of the potential and of its normal derivative, respectively, for points on the surface of the vertical cylinder separating the near field \( D_0 \) from the far-field \( D_{D_N} \). The continuity of the potential is obtained from Eqs. (6.2) and (6.3) by letting \( r \in \partial D_C \):

\[
\int_{\partial D_0} \sigma_N(r_0) G_N(r; r_0) dS(r_0) - \sum_{j=1}^{n} \sigma(r_j) G(r; r_j) S_j = 0. \tag{6.7}
\]

From the integral representations (6.2) and (6.3), we obtain corresponding expressions for the derivatives of the corresponding potentials, which permit us to express finally the continuity of the normal derivative of the potential for points on the vertical cylinder \( (r \in \partial D_C) \) as follows:

\[
-\frac{\sigma_N(r)}{2} + \int_{\partial D_0} \sigma_N(r_0) \frac{\partial G_N(r; r_0)}{\partial n} dS(r_0)
- \sum_{j=1}^{n} \sigma(r_j) \frac{\partial G(r; r_j)}{\partial n} S_j = 0, \quad r \in \partial D_C. \tag{6.8}
\]

Thus, the hybrid scheme is finally formulated as an integral equation on the various parts of the boundary of the interior domain \( \partial D_0 \), and consists of (i) Eq. (6.4) on the body surface and the bottom surface, respectively, (ii) Eq. (6.5) on the free-surface, and (iii) Eqs. (6.7) and (6.8) on the vertical cylinder. The forcing of the various subproblems comes from the Neumann data on the wetted surface of the floating body, as defined by Eqs. (6.6) for the various subproblems (diffraction and radiation potentials). Again, we concentrate ourselves on the solution of the above system for regular frequencies only.

It is worth noticing here that the presence of the bottom-dependent Green’s function \( G \) (studied in detail in Ref. [25]) in the expansion (6.3), offers us a consistent modelling of the dynamics of the examined hydrodynamic problems at infinity. This fact is very important concerning the behaviour of floating bodies over an uneven bottom topography, with different depths at infinity as approached from different directions (i.e. different azimuthal angles \( \theta = \tan^{-1}(y/x) \) on the horizontal plane). As an example, we present in Fig. 7 the horizontal wave pattern associated with this Green’s function, located a small depth below the free surface, in the case a of a shoal characterised by monotonically decreasing depth and in the case of the same shoal as before with a ridge at the middle. We can see from Fig. 7(a) that, in the case of monotonic bed profile, the main characteristics of the far-field are the formation of a shadow zone with an ever expanding width, which is located along the bottom irregularity \( (\theta = 0, \pi) \), and that in each of the two sectors not including the bottom irregularity \( (0 < \theta < \pi \text{ and } \pi < \theta < 2\pi) \) the asymptotic behaviour of the wave field approaches the form of an outgoing cylindrical wave propagating with wavelength corresponding to the sector-depth at infinity.

This situation could change completely in the case of a non-monotonic bottom profile of the form of a smooth underwater ridge, as the second profile examined in Fig. 7(b). If the position of the source is taken to be near the top of the ridge, an energy duct could be formed along the bottom irregularity, as it is clearly illustrated. In this case, the wave energy penetrating in the duct is trapped, and the strength of the wave field does not attenuate with range (from the source) inside the duct.

### 6.1. Simplified version of the 3D hybrid scheme

As already mentioned, a major difficulty associated with the solution of Eqs. (6.4)–(6.8) deals with the calculation of the bottom-dependent Green’s function \( G(r; r_0) \). Details about this calculation are provided in Belibassakis and Athanassoulis [25]. The previous task could become quite demanding computationally, especially when considering the contribution of many such sources (i.e. when \( J \) in the right-hand side of Eq. (6.3) is large). Simplified versions of the previous hybrid scheme can be obtained by taking the radius of the vertical cylinder \( \rho_C \) to be large \( (\rho_C \to \infty) \), and in practice, extending several wavelengths away from the floating body and the bottom irregularity. In the case of only one floating body, if \( \rho_C \) is assumed to be quite large, then only one term in Eq. (6.3) is sufficient, and the summation in the right-hand side can be truncated to \( J = 1 \), corresponding to only one

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**Fig. 7.** Horizontal pattern of the bottom-dependent source over (a) a shoal with monotonically varying depth and (b) an underwater ridge superimposed to a shoal.
bottom-dependent source (located at \( r_{0,1} \)). For bodies of simple geometry, a good choice for the position of the source \( r_{0} \) is the Centre of Buoyancy associated with the immersed volume. In this case, the unique coefficient \( a(r_{0,1}) \) can be eliminated from Eqs. (6.7) and (6.8), resulting in
\[
\frac{\partial c(r)}{\partial t} - U(r)c(r) = 0, \quad r \in \partial D_c,
\]
where the function \( U(r) \) is defined as the ratio of the normal derivative to the value of the bottom-dependent source (calculated for points lying on the vertical cylinder):
\[
U(r) = \frac{1}{d(r_{0,1})} \frac{\partial c(r_{0,1})}{\partial n}, \quad r \in \partial D_c.
\]

6.2. Numerical scheme: accuracy and convergence of the numerical solution

Before proceeding to the discussion of examples concerning scattering by 3D floating bodies in variable bathymetry regions, in this subsection we provide details about the mesh generation on the various parts of the boundary and present comparisons of our results with benchmark solutions in cases concerning similar 3D problems in deep water. We note here that in order to obtain numerical solution in deep-water conditions by the present method it suffices to consider the bottom boundary surface at a depth greater than half the characteristic wavelength.

In all examples considered herein, the grid is obtained by matching the subgrids generated on the various parts of the boundary. The hybrid scheme, has been materialised by means of a low-order boundary element technique, based on 4-node quadrilateral (hyperboloid) elements for approximating the geometry, ensuring continuity (no-gaps) between elements, and piecewise constant distribution of the simple (Rankine) sources in each element (see, e.g., Ref. [33]).

As an example, in Fig. 8(a) and (b) the grid on a 3D floating ellipsoid is presented, by using 20 elements along the half cross-section and 45 elements along the waterline. The grid on the body surface is then matched with the corresponding one on the free surface, by using a grid of 45 elements along radius and 45 elements along the circumference. Also, the free-surface grid is matched with the one on the outer cylindrical surface, by using 45 elements along the azimuthal direction and a number of elements (here 5) along the vertical direction. Finally, the grid on the bottom surface is similar to the one on the free surface (and consists of 45 \( \times 45 \) elements). Thus, in this case the total number of elements is 5175.

In Fig. 9, we present comparisons concerning the added-mass and damping coefficients in the case of a floating hemisphere (of radius \( a \)), in surge and heave motions, as obtained by the present method and by using complete expansions of the wave potentials in spherical harmonics, Hulme [37]. Following the latter work, the normalisation used for the added-mass and damping coefficients plotted in Fig. 9 is \( (a_{ah}/(2\pi/3)pa^3) \) and \( (b_{ah}/(2\pi/3)pa^3) \), respectively. In the same figure, the convergence of the present method is illustrated by showing, except of the results obtained using the grid based on 20 \( \times 45 \) elements on the body surface and 5175 totally (as described before), also the numerical results obtained by using a second grid (of about half size in comparison to the previous one), based on a discretisation of 15 \( \times 32 \) elements on the body surface and 2688 totally on the boundary surface.

A similar comparison is presented in Fig. 10 concerning a floating ellipsoid with axes ratio 1.025:0.25:1 in surge and pitch motions, as obtained by the present method and by solving the integral equation on the body boundary surface, using the deep-water Green's function [38]. In this case, the normalisation used for the added-mass and damping coefficients obtained in Fig. 9 is \( (a_{ah}/a^3) \), \( (b_{ah}/a^3) \), respectively. In both cases examined in Figs. 9 and 10, we are able to observe the convergence of the present method to the benchmark solution(s) with increasing number of elements, as well as the good accuracy achieved by using a grid size of the order of 5000 boundary elements. It is worth noticing here that due to the simplicity of the Rankine source distribution the coefficient matrix is computed very fast and the computation time essentially depends on the solution of the final linear system. As an indication of the computation time, the calculation of excitation (propagation and diffraction) forces and all hydrodynamic coefficients \( a_{hl}, b_{hl} \) at one frequency, in the case of the grid with 5175 elements, requires 2 CPU min on a PC-AMD dual core 2.21 GHz processor.

6.3. Numerical results concerning floating bodies in variable bathymetry

As an application of the present theory to floating bodies in variable bathymetry, a numerical example is examined and discussed in this subsection concerning the diffraction problem of a 3D body over a smooth shoal. For simplicity, we consider an oblate spheroid (with 3 planes of symmetry) of radius 5 m and draft 1 m, floating at its horizontal plane of symmetry, as shown in Fig. 11. The floating body is lying over a smooth, but locally steep shoal, characterised by the following depth function:
\[
h(y) = \frac{h_1}{2} + \frac{h_3}{2} - \frac{h_1 - h_3}{2} \tanh(0.05y), \quad -\infty < y < \infty.
\]
In this environment, the depth ranges from $h_1 = 4\text{m}$ to $h_3 = 2\text{m}$, and the maximum bottom slope (at $y = 0$) is almost 50%. We also consider an obliquely incident harmonic wave of period $T = 3\text{s}$, propagating with direction $45^\circ$ with respect to the parallel bottom contours (indicated by using an arrow in Fig. 11). In this case, the characteristic wavelength varies from $\lambda_1 = 13.36\text{m}$ to $\lambda_3 = 11.3\text{m}$ and thus, the shoaling ratio ranges from $h_1/\lambda_1 = 0.3$ to $h_3/\lambda_3 = 0.17$, implying intermediate-depth wave conditions everywhere in the flow domain.

In Fig. 11(a), the generalised incident wave field over the above smooth shoal is plotted on the horizontal plane without considering the effect of the floating body. The field is shown by
using equipotential lines, as calculated by means of the present coupled-mode model, described in Section 3 and appropriately extended to treat obliquely incident waves (see Ref. [24]). The location of the floating body is indicated both on the horizontal and the vertical planes. In this case, the calculated values of the reflection and transmission coefficients are $A_R = 0.0054$, $AT = 0.909$, and the direction of wave propagation with respect to the bottom contours in the region of transmission is calculated to be 36.6°. For comparison, in Fig. 11(b) the same result is shown, but for a uniform strip of constant depth equal to the average water depth ($h = 3\, \text{m}$). In this case, the direction of wave propagation remains 45° everywhere.

On the basis of the calculated generalised incident wave field, the Neumann data on the solid surface of the floating oblate spheroid are calculated by using Eq. (6.6a). Then, the diffraction problem is solved by means of the present simplified hybrid scheme consisting of Eqs. (6.4), (6.5) and (6.9). In the examined case, the radius of the vertical cylinder

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**Fig. 10.** Added-mass and damping coefficients vs. non-dimensional wavenumber in the case of a floating ellipsoid in heave and pitch motions, in deep water. Comparison of the present method results using $15 \times 32$ elements on the body surface (indicated by stars) and $20 \times 45$ elements (indicated by circles) vs. the solution obtained by Kim [38] shown by using solid and dashed lines, respectively.

**Fig. 11.** Real part of the generalised incident wave field $\phi_P$ over the smooth shoal, without the effect of the 3D floating spheroid. (a) The case of the shoal defined by Eq. (6.14). The depth ranges from 4 to 2 m and the depth-contours are shown on the horizontal plane by using dashed lines. (b) The case of a constant-depth strip ($h = 3\, \text{m}$).
separating the inner from the outer domain has been selected to be \( r_c = 25 \text{ m} \) (i.e. 5 times greater than the radius of the floating body), lying almost 2 wavelengths away from the floating body and bottom irregularity for this wave frequency \( (\omega = 2.09 \text{ rad/s}) \). Numerical experimentation has shown that in the examined case such distance of the vertical cylinder from the

![Fig. 12. Left column: Real part, imaginary part and modulus of the calculated diffraction potential on the free surface \((z = 0)\) around an oblate spheroid floating over the shoal \((6.14)\). Right column: The same result but for the floating oblate spheroid in a constant-depth strip \((h = 3 \text{ m})\).](image-url)
inhomogeneity is quite enough to guarantee the convergence of our calculations.

In Fig. 12, we present numerical results concerning the diffraction field associated with the 3D oblate spheroid, lying over the variable bathymetry (left column of subplots). For comparison, in the same figure (right column of subplots) corresponding results are shown for the same floating body in the constant-depth strip \( h = 3 \, \text{m} \), as also obtained by the present method. The boundary mesh is similar to the one presented in Fig. 8 and consists by 5175 quadrilateral elements totally, distributed as follows: 900 elements for discretising the body surface and 2025 elements for each one of the free surface and bottom surface, and 225 for discretising the outer cylindrical surface.

Moreover, the present method seems to correctly model the 3D features of the reflected–diffracted wave field in the example considered. In particular, in Fig. 12, and especially from the last row of subplots concerning the modulus of the potential, we observe that in both cases (shoal, constant-depth strip) the diffracted field is much stronger in the downwave direction (shown by using solid arrows), than in the upwave direction (shown by using dashed arrows). Furthermore, in the case of constant-depth strip the downwave direction is 45°, exactly opposite of the direction of the upwave diffracted (reflected) wave, as expected due to symmetry. However, in the case of the floating body over the shoal the direction of the downwave diffracted wave is about 32°, and a corresponding change can also be observed from the left column subplots of Fig. 12 concerning the reflected wave (propagating in the upwave direction). An explanation of the observed changes concerning the direction of the diffracted wave over the shoal is the effects of refraction, that are induced by the shoaling (depth variation) which modifies the preferred directions of wave radiation in the regions of wave incidence \( (D^{13}) \) and wave transmission \( (D^{17}) \), respectively.

7. Conclusion

A numerical technique, based on the coupled-mode theory by Athanassoulis and Belibassakis [23] for the propagation of water waves over variable bathymetry regions, in conjunction with a hybrid method based on boundary integral representations of the near-flow and suitable representations of the far-flow dynamics, has been developed to treat the problem of hydrodynamic analysis of floating bodies in general bathymetry. The present method is free of any mild-slope assumption concerning the seabed topography. Both 2D and 3D problems have been considered. In all cases the near field is represented by boundary integral representation involving simple (Rankine) sources. In the 2D case, the far-field is modelled by complete (normal-mode) series expansions derived by separation of variables in the constant-depth half-strips. In the 3D case, the far-field is modelled by an integral representation involving the appropriate Green’s function for harmonic water waves over a bottom with different depths at infinity, developed by Belibassakis and Athanassoulis [25]. The present numerical solution is obtained by a low-order panel method. Numerical results are presented concerning floating bodies of simple geometry, lying over sloping and undulating seabeds. In the case of 2D floating pontoon the effects of bottom slope and curvature on the hydrodynamic characteristics (hydrodynamic coefficients and responses) of the floating bodies are examined, showing that bottom variations could significantly influence the responses of the floating body with respect to all degrees of freedom. With the aid of comparisons with benchmark solutions, the convergence and accuracy of the present method in 3D has been studied. As a first application of the present theory to variable bathymetry, a numerical example is presented and discussed concerning the diffraction problem of a 3D floating body over a smooth shoal. Although, in this case, the physical properties of the medium (the index of refraction and the propagation speed) are different at infinity as approached from different directions, and the far-field wave pattern is not known, the present method seems to correctly represent the 3D features of the reflected–diffracted wave field.

More detailed results concerning 3D radiation problems in general bathymetry and comparisons with other numerical methods, as well as with experimental data, aiming to further validation of the present method, is left to be subject of a future work. The latter will also focus on the calculation of hydrodynamic responses and wave loads of 3D bodies in general bottom topography and the extension of the present theory to more complex geometric configurations including multiple floating bodies.

References