A novel coupled-mode theory with application to hydroelastic analysis of thick, non-uniform floating bodies over general bathymetry

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Abstract: A new coupled-mode system of horizontal equations is presented for the hydroelastic analysis of large floating bodies or ice sheets of general, finite thickness, lying over variable bathymetry regions. The present method is based on the theory of shear deformable plates (or beams), and is derived by an enhanced representation of the elastic displacement field, containing additional elastic vertical modes and permitting shear strain and stress to vanish on both the upper and lower boundaries of the thick floating plate. This model extends existing third-order plate theories to plates and beams of general shape. The presented coupled-mode system of horizontal differential equations is obtained by means of a variational principle composed of the one-field functional of the elastodynamics in the plate region, and a pressure functional in the water region. The wave potential in the water column is represented by means of a local-mode series expansion containing an additional mode providing the appropriate correction term on the bottom boundary, when the slope is not mild. In the above sense, the proposed method extends previous approaches concerning hydroelastic problems based on thin-plate theory. The focus of this work is on the scattering of linear, coupled, hydroelastic waves propagating through an inhomogeneous sea ice environment, containing ice sheets of variable thickness and a non-mildly-sloped interface. Numerical results are presented in the simple two-dimensional case, showing that the presented approach efficiently models the hydroelastic problem and is able to provide accurate results when only a few terms are used in the expansion. Ideas for extending the proposed method to three dimensions are also discussed.

Keywords: hydroelastic analysis, thick ice sheets, VLFS, variable bathymetry, coupled-modes

1 INTRODUCTION

The effect of water waves on large floating deformable bodies is an interesting problem that is receiving increasing attention because of its applicability to environmental issues. A specific example concerns the interaction of waves with thin sheets of sea ice, which is particularly important in the marginal ice zone (MIZ) in the Antarctic, a region consisting of loose and packed ice floes that is situated between the ocean and the shore sea ice (see, e.g. Porter and Porter [1]). Since ice sheets can support flexural–gravity waves, the energy carried by the ocean waves can be propagated far into the MIZ, contributing to the break-up and melting of the ice glaciers [2, 3] thereby accelerating global warming effects and the rise of sea water levels. Another example deals with the analysis of very large floating structures (VLFS) and platforms of shallow draft, cases where hydroelastic effects become significant. Such structures have been intensively studied, and are being considered for use as floating airports and mobile offshore bases (see, e.g. Kashiwagi [4], Watanabe et al. [5], and Jin and Xing [6]). An extended, recent review on both topics and the

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synergies between VLFS hydroelasticity and sea ice research can be found in Squire [7].

General methods for the dynamical analysis of such complex fluid–elastic structure interaction problems have been developed, based on variational principles and finite element modelling (FEM) (see, e.g. Xing et al. [8]). Taking into account that the horizontal dimensions of the large floating body are much greater than the vertical one, thin-plate (Kirchhoff) theory is commonly used to model this situation. Also, although non-linear effects are of specific importance (see, e.g. Faltinsen [9]), the solution of the linearized problem still provides valuable information, serving also as the basis for the development of weakly non-linear models [10].

The linearized hydroelastic problem is effectively treated in the frequency domain, and many methods have been developed for its solution. These include hydroelastic eigenfunction expansion techniques [11–13], the boundary element method (BEM) [14, 15], B-spline Galerkin method [16], integro-differential equations [17], Wiener–Hopf techniques [18], Green–Naghdi models [19], Galerkin methods [20], variational equations [21], and others. In addition to the above, high-frequency asymptotic methods have been developed to describe the deflection dynamics of VLFS, see, e.g. Ohkusu and Namba [22] and Hermans [23]. The latter are especially useful in the case of short waves interacting with a floating structure with large horizontal dimensions.

Similar techniques have been developed for the interaction of water waves with ice sheets. For example, Marchenko and Shrira [24], using Zakharov’s variational principle [25], developed a Hamiltonian formalism for the waves in the water beneath an ice sheet, and Meylan and Squire [26] used a Green’s function approach to formulate an integral equation over the floating plate. In the case of an ice sheet of varying thickness, floating on water of varying depth, Porter and Porter [1] derived a simplified model using a variational principle and invoked the mild-slope approximation with respect to the ice thickness and water depth variations. Numerical methods for predicting the linearized hydroelastic responses of VLFS in variable bathymetry regions have also been developed, based on the BEM [27, 28], FEM [29], on eigenfunction expansions in conjunction with step-like bottom approximation [30]. In the case of the hydroelastic behaviour of large floating bodies over general bathymetry, modelled as thin homogeneous plates, a coupled-mode system has been derived and examined by the current authors (Belibassakis and Athanassoulis [31]). This method is based on a local vertical expansion of the wave potential in terms of hydroelastic eigenmodes, extending previous similar approaches for the propagation of water waves in variable bathymetry regions [32, 33]. A similar approach based on multi-mode expansion has been presented by Bennets et al. [34], with application to wave scattering by ice sheets of varying thickness.

In this work attention is focused on a hydroelastic analysis of a thick, non-uniform, shear deformable floating elastic body, lying over a variable bathymetry region. The problem addressed is the scattering of linear, coupled, hydroelastic waves propagating through an inhomogeneous sea ice environment, containing ice sheets of variable thickness and non-mildly-sloped interface. A new coupled-mode system of horizontal equations is presented, based on the theory of shear deformable plates (or beams), derived by an enhanced representation of the elastic displacement field. The presented model contains additional elastic vertical modes, permitting the shear strain and stress to vanish on both the upper and lower boundaries of the thick floating plate. This model extends third-order plate theories by Reddy [35] and Bickford [36] (see also Wang et al. [37]) to plates and beams of general shape. The present coupled-mode system of horizontal differential equations is obtained by means of a variational principle composed by the one-field functional of elastodynamics (see, e.g. Graff [38] and Reddy [39]) in the plate region, and the Luke’s pressure functional in the water region [40]. The wave potential in the water column is represented by means of a local-mode series expansion containing an additional mode providing an appropriate correction term on the bottom boundary, when the slope is not mild. In the above sense, the present method extends previous approaches concerning free-surface hydrodynamic and hydroelastic problems based on thin-plate theory [1, 10, 31].

The rest of this paper is organized as follows. In section 2 the mathematical formulation of the problem is presented and a variational principle is given, based on the stationarity of a functional composed by the one-field functional associated with Hamilton’s principle in linear elastodynamics, in conjunction with Luke’s functional, based on the integration of pressure in the vertical water column. Then, in section 2.2 an enhanced, local-mode representation for the time-harmonic wave potential in the water region is introduced based on hydroelastic eigenmodes. After a short introduction to first- and higher-order plate/beam theories, the
enhanced vertical expansion of the elastic displacement field in the floating elastic plate of finite and general thickness is presented in section 3.2, as well as the representation of the strain tensor components. The above representations used in the variational principle lead finally to a novel coupled-mode system (CMS) of horizontal equations, with respect to the mode amplitudes of the wave potential and the elastic plate modes, discussed in section 4.

Subsequently, by studying spatially periodic solutions, an enhanced hydroelastic relation is derived that carries the effects of finite floating plate thickness, shear ratio, and thickness to wavelength ratio. The effects of these parameters on the phase speed of hydroelastic waves in terms of the shoaling ratio are presented and discussed in section 4.3. Numerical examples are presented in section 4.4, illustrating the effect of variable bathymetry and variable plate thickness on the solution. With the aid of specific comparisons, the effects of bottom and floating plate slope on the hydroelastic response of large, shear deformable, floating elastic plates of finite thickness are examined and discussed. Finally, in section 5 a simplified model of the presented CMS is derived, governed by the enhanced hydroelastic relation. This approximation is based on a slowly varying plate thickness, keeping the effects of general depth without any further assumptions concerning the mildness of bottom slope. The simplified system facilitates the modelling of floating, shear deformable bodies, with variable thickness, elastic parameters, and mass distribution, including applications to the problem of wave interaction with ice sheets of general morphology.

Future work is planned towards the fully three-dimensional (3D) problem concerning finite rafts over a general seafloor, and the examination of the effects of non-linearity.

2 FORMULATION OF THE PROBLEM

The studied environment consists of a water layer bounded above by a floating plate of general shape, i.e. an ice sheet of variable thickness \( b(x) \) or a large platform, and below by a rigid bottom; see Fig. 1. For simplicity in the presentation, the upper face of the floating elastic plate is flat-horizontal (e.g. by considering the static plate deflection to be very small), however, the presented analysis can be easily extended to the case of a more general upper boundary. It is also assumed that the bottom and plate surfaces exhibit general one-dimensional variation in a subdomain of finite length. The bathymetry is characterized by straight and parallel bottom contours lying between two regions of constant but possibly different depth: \( h = h_1 \) (region of incidence) and \( h = h_3 \) (region of transmission), where \( h(x) \) is the depth function. A Cartesian coordinate system is introduced, with its origin at some point on the upper elastic-plate surface (in the variable bathymetry region), the \( z \)-axis pointing upwards and the \( y \)-axis being parallel to the bottom contours. The functions \( h(x) \) and \( b(x) \) are smooth functions such that \( h(x) = h_1 \) and \( b(x) = b_1 \), for all \( x \leq x_1 \), and \( h(x) = h_3 \) and \( b(x) = b_3 \), for all \( x \geq x_3 \).

Also, the slope of the elastic-plate deflection \( u(x,y,t) \)
is assumed small, so that linear theory can be applied.

The scattering problem of harmonic incident plane waves of angular frequency \( \omega \), under the combined effects of variable bathymetry and the infinite, floating elastic plate of general and finite thickness \( b \) is addressed. Attention is focused on the case of normally incident linear waves (as shown in Fig. 1), leaving the treatment of obliquely incident waves propagating with directions \( \theta_1 \) and \( \theta_2 \) with respect to the \( x \)-axis in the regions of incidence \( (x < x_1) \) and transmission \( (x \geq x_3) \), respectively, and more complex 3D systems to future work.

The above problem has been considered by many authors in the framework of thin-plate (Kirchhoff or classical plate) theory, under the assumptions of a small uniform thickness and constant depth (see, e.g. Squire [7]). Under the same assumption concerning the plate characteristics, Belibassakis and Athanassoulis [31] developed a coupled-mode model for examining the effects of a variable bottom topography, but for a floating body of limited horizontal extent. Moreover, Porter and Porter ([1], under the mild-slope hypothesis), and Bennetts et al. ([34], using a multi-mode expansion) presented similar models in the case of an infinite floating ice sheet of variable characteristics (e.g. variable thickness).

The present work is a generalization of the authors’ coupled-mode model in general bathymetry, for the hydroelastic analysis of floating, shear deformable plate or ice sheet, characterized by possibly non-mildly-sloped boundaries, supporting the study of finite and general thickness effects. It is based on enhanced representations of both the elastic displacement field in the plate region and the wave potential field in the water region, allowing the modelling of shear stress effects and the consistent treatment of the end-conditions on the non-horizontal boundaries (wetted plate surface, bottom surface).

### 2.1 Variational formulation

Under the assumptions of incompressibility and irrotationality, the problem of the evolution of water waves propagating over a variable bathymetry region allows not only Hamiltonian-type formulations, constrained on the below-the-surface kinematics [24] but also unconstrained variational formulations, as e.g. the one proposed by Luke [40]. The variational formulation of the present problem is based on a functional \( \mathcal{F} = \mathcal{F}_1 + \mathcal{F}_2 \), composed by the one-field functional \( \mathcal{F}_1 \) associated with Hamilton’s principle in linear elastodynamics (see, e.g. Graff [38] and Reddy [39]), and Luke’s functional \( \mathcal{F}_2 \) formed by integrating the pressure in the vertical water column.

The former functional is defined through \( K - U + f_b \), including the difference between the kinetic and elastic energy and the gravity (body) forces \( f_b = -\rho_g \mathbf{k} \mathbf{u} \), where \( \mathbf{k} \) denotes the unit vector in the direction of the vertical \( z \)-axis. The difference \( K - U \) can be analysed as follows.

1. The kinetic energy density

\[
K = \frac{1}{2} \rho \left( \frac{\partial \mathbf{u}}{\partial t} \right) \cdot \left( \frac{\partial \mathbf{u}}{\partial t} \right)
\]

where \( \rho \) is the elastic material density and \( \mathbf{u} = (u_x, u_y, u_z) = (u_x, u_y, 0, u_z) \) is the displacement vector field in the elastic region \( D_E = (-\infty, \infty) \times (-b(x) + w_t, 0) \).

2. The elastic energy density is \( U = \sigma_{ij} \varepsilon_{ij}/2 \), where \( \sigma_{ij} \) denotes the stress and \( \varepsilon_{ij} \) the strain tensor field, which satisfies Cauchy equations \( \varepsilon_{ij} = (u_{i,j} + u_{j,i})/2 \).

In the above formulae and in the following discussion the standard tensor convention, concerning the summation of repeated indices, is used.

Finally, \( \mathcal{F}_1 \) is obtained by integrating over all the elastic domain and in time

\[
\mathcal{F}_1 = \int_{t_1}^{t_2} \int_{D_E} \int dz \, dx \, dt \left[ \frac{1}{2} \rho \left( \frac{\partial \mathbf{u}}{\partial t} \right)^2 - \frac{1}{2} \sigma_{ij} \varepsilon_{ij} - \rho_g \mathbf{k} \mathbf{u} \right]
\]

(1)

In the present problem only normal stresses on the wetted boundary are considered by the action of water pressure. As stated at the beginning of section 2 the time-independent component of elastic deformation and stresses are considered to be very small and thus, in this study attention is focused only on the dynamical part. Otherwise, the effects of static deflection should be included in the definition of the geometrical configuration and the dynamical equations will be finally obtained by subtracting by parts the equations concerning the total (static and dynamic) field from the ones concerning the static field only.

Moreover, taking into account the stress–strain relations in their standard form for an isotropic elastic material
where \( \Delta = e_{kk} \), \( \delta_{ij} \) is Kronecker’s delta, and \( \mu = G \), \( \lambda = \nu E/(1-2\nu)(1+\nu) \) the Lamé constants, expressed through the Young’s modulus \((E)\), shear modulus \((G)\), and Poisson’s ratio \((\nu)\), the functional \( F_1 \) is finally expressed in terms of \( u_i \). By defining the surface field

\[
 w(x; t) = u_z(x, z = -b(x); t) 
\]

(3)

to be the vertical motion of the lower elastic plate boundary, \( F_1 = F_1 [u_i; w] \) becomes a functional of the spatial \((u_i)\) and surface \((w)\) elastic fields.

The second term in the definition of the above composite functional is defined as follows

\[
 F_2 = \int_{D_N} \int P \, dz \, dx \, dt 
\]

(4)

where \( D_N = (-\infty, \infty) \times (-h(x), -b(x) + w) \) is the water region, and \( P \) denotes the pressure obtained through Bernoulli’s equation

\[
 P = -\left\{ \rho \frac{\partial \Phi}{\partial t} + \frac{1}{2} \rho c^2 \left( \frac{\partial \Phi}{\partial x} \right)^2 + \frac{1}{2} \rho c^2 \left( \frac{\partial \Phi}{\partial z} \right)^2 + \rho g (z - z_i) \right\} 
\]

(5)

where \( \rho \) is the water density and \( \Phi(x, z; t) \) denotes the velocity potential. Thus, \( F_2 = F_2 [\Phi; w] \) is a functional on the spatial wave potential and surface \((w)\) fields. In equation (4), \( z_i \) is a correction factor introduced to account for the position of the mean water level (shown in Fig. 1 by using a dashed line), which in the case of a homogeneous floating plate or sheet of uniform density \((\rho_E)\) is \( z_i = (\rho_E/\rho - 1)b \).

The present variational formulation is completed by calculating the first variation of the above functional and requiring its stationarity, that is

\[
 \delta F = \delta F_1 [u_i; w] + \delta F_2 [\Phi; w] = 0 
\]

(6)

where the variation of each term is obtained as the sum of partial variations with respect to all involved fields. The above equation, after simplifications based on the smallness of \( w \) and \( V \Phi \) associated with linearization, using standard arguments of the calculus of variation and Green’s theorem, and considering the variations to vanish at infinity \((x \to \pm \infty)\) and at the ends of the time interval, finally results in

\[
 \begin{align*}
 \int_{t_1}^{t_2} \int_{z=0}^{z=b(x)} \frac{\partial}{\partial t} \left[ p_E \left( \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial z^2} \right) + \rho g \delta \Phi \delta u_i \right] \, dx \, dz \\
 + \int_{x}^{t_1} \int_{z=0}^{z=b(x)} \frac{\partial}{\partial x} \left[ \sigma_{ij} \left( \frac{\partial^2 u_i}{\partial x^2} - \frac{\partial^2 u_i}{\partial z^2} \right) + \rho g \delta \Phi \delta u_i \right] \, dx \, dz \\
 + \int_{x}^{t_1} \int_{z=0}^{z=b(x)} \frac{\partial}{\partial z} \left[ \sigma_{ij} \left( \frac{\partial^2 u_i}{\partial x^2} - \frac{\partial^2 u_i}{\partial z^2} \right) + \rho g \delta \Phi \delta u_i \right] \, dx \, dz \\
 + \int_{x}^{t_1} \left[ \rho \delta \Phi \delta w + g(-b + w - z_i) \delta w \right] \, dx \\
 - \rho \int_{x}^{t_1} \left[ \delta \Phi \frac{\partial w}{\partial x} + \delta \Phi \frac{\partial w}{\partial z} \right] \, dx \\
 - \rho \int_{x}^{t_1} \left[ \delta \Phi \frac{\partial w}{\partial x} + \delta \Phi \frac{\partial w}{\partial z} \right] \, dx \\
 = 0 
\end{align*} 
\]

(7)

where \( n_i \) denotes the outward unit normal vector (i.e. \( n_i = \hat{k} \) on the upper elastic boundary \( z = 0 \), and \( n_i = -\left(b_x, -1\right)/\sqrt{1 + \left(b_x\right)^2} \) on the lower one \( z = -b(x) \). Here, and in the following, a prime denotes differentiation with respect to the argument appearing in the subscript. Details concerning the calculation of \( \delta_u F_1 \) can be found in standard textbooks (see, e.g. Graff [38]). Details concerning the calculation \( \delta_u F_2 [\Phi, w] = \delta_u F_2 + \delta_u F_1 \) can be found in Witham [41]. More precisely, the variational equation \( \delta_u F = 0 \) models the water wave kinematics (under the floating elastic plate), and \( \delta_w F_2 \) the water wave dynamics.

The differential formulation of the problem is obtained from equation (7) using the independence of all involved variations. The first term of the left-hand side of equation (7) leads to the equations of linear isotropic elasticity in \( D_E \). The second and third terms by themselves require the balance of normal and shear stresses on the elastic plate boundaries. On the upper flat horizontal boundary the second term leads to the following two conditions necessitating zero normal and shear stresses

\[
 \sigma_{xx} = \sigma_{zz} = 0, \quad \text{on} \quad z = 0 
\]

(8)
On the lower elastic plate surface the third term of equation (7) associated with the variation \( \delta u_x \) of the horizontal displacement field leads to

\[
-b'_x \sigma_{xx} - \sigma_{xz} = 0, \quad \text{on} \quad z = -b(x) \tag{9}
\]

requiring vanishing tangential shear stresses on the lower boundary. On the other hand, the third term of equation (7) associated with the variation \( \delta w \) of the vertical displacement on the lower boundary combines with the fourth term of equation (7), coming from the variation \( \delta w_i F_z \) and leads finally to the condition

\[
\sigma_{zz} - (b'_x) \sigma_{xz} = -\rho \left[ \left( \frac{\partial \Phi}{\partial t} \right) \bigg|_{z = -b} + g \, w \right],
\]

on \( z = -b(x) \) \tag{10}

noticing that the gravity body forces (mass of floating sheet) in the first term \( (\rho g b) \) is cancelled out by the hydrostatic buoyancy component \( (\rho (b + z)) \) in the fourth term. In completing the presentation of the above variational principle, its fifth term leads to the Laplace equation in the water domain, and finally, the sixth and seventh terms are the kinematical conditions on the wetted moving boundary and on the impermeable bottom, respectively, both of which are not necessarily restricted to be mildly-sloped surfaces.

The above unconstrained variational formulation of the hydroelastic problem allows the introduction of appropriate specific representations of the unknown elastic displacement and wave potential fields (e.g. consistent vertical expansions of these fields in the elastic and water domains, respectively), and the derivation of novel sets of coupled equations governing the considered hydroelastic system. The unknown fields in these new equations are the coefficients of the vertical expansions, which are simply the horizontal mode amplitudes. In this way, a kind of reduction of dimensionality is achieved, by separating the horizontal space where propagation and other wave phenomena manifest and where also the mode amplitudes are defined, from the cross (vertical) space. The second important aspect of the present formulation is that it can be directly extended to 3D problems and that it can be generalized to treat the non-linear problem associated with finite amplitude waves, by keeping all terms entering in the variational principle (7) without making the simplifications based on the smallness of \( w \) and \( V\Phi \).

### 2.2 Vertical expansion of the wave field in the water region

For the case of harmonic incident linear waves, the wave potential, as well as other involved fields, allow standard representations of the form \( \Phi(x, z; t) = \text{Re}(\psi(x, z) e^{-i\omega t}) \) to be written. The vertical structure of the complex wave potential \( \psi(x, z) \) in the general bathymetry region is represented by a local-mode series of the form

\[
\psi(x, z) = \psi_{-1}(x) Z_{-1}(z; x) + \sum_{n=0}^{\infty} \psi_n(x) Z_n(z; x),
\]

where \( \psi_{-1}(x) \) is an appropriate term, called the sloping-bottom mode, that accounts for the satisfaction of the bottom boundary condition on the non-horizontal parts of the bottom. The idea of the sloping bottom mode, in conjunction with the above type of modal expansion, was first introduced by Athanassoulis and Belibassakis [32] and Belibassakis et al. [33] for the case of water waves propagating in a variable bathymetry. It has since been applied to various problems that exhibit similar features, such as non-linear water waves [42–44], hydroacoustics [45], and hydroelastic applications in variable bathymetry regions, formulated in the context of thin-plate theory [31]. In accordance with the latter work, the infinite set \( Z_n(z; x) \), \( n = 0, 1, 2, 3, \ldots \), of functions describing the vertical structure of each mode, at each horizontal position \( x \), are generated by a local, vertical, Steklov-type eigenvalue problem of the form

\[
\frac{d^2 Z_n(z)}{dz^2} - \kappa_n^2 Z_n(z) = 0,
\]

in the interval \( -h < z < 0 \) \tag{12}

\[
\frac{dZ_n(z = -h)}{dz} = 0, \quad \text{at the bottom} \quad z = -h \tag{13a}
\]

\[
\alpha(\kappa) \frac{dZ_n(z = -b)}{dz} - \mu Z_n(z = -b) = 0,
\]

at the interface \( z = -b(x) \) \tag{13b}

where \( \mu = \omega^2 \rho / g \) is the frequency parameter. The specific form of the function \( \alpha \) is

\[
\alpha(\kappa) = D e \kappa^4 (1 - \delta) + 1 - \kappa^2 \frac{b^2}{12} (1 - \delta) \tag{13c}
\]
as will be shown in section 4.3, where $D$ denotes the flexural rigidity coefficient and $e$ the mass coefficient of the plate (see section 4.3), respectively, and $e$ is a material constant defined later on by equation (30). The solution of the problem constituted by equations (12) and (13) is given by

$$Z_n(z) = \frac{\cosh[k_n(z + h)]}{\cosh(k_n h)}, \quad n = 0, 1, 2, 3, \ldots$$  \hspace{1cm} (14)

where the eigenvalues $\{k_n, n = 0, 1, 2, \ldots\}$ are obtained as the roots of the (local) dispersion relation

$$\mu H = \sigma(\kappa) \kappa H \tanh(\kappa H)$$  \hspace{1cm} (15)

In the general case, the involved parameters $\kappa$ and $H$ in equation (15) are considered to be functions of $x$. The expression (13c) for $z(\kappa)$, leads to a perturbed version of the standard hydroelastic dispersion relation based on thin-plate theory (see, e.g. equation 4.5c in Belibassakis and Athanassoulis [31]), which is

$$\mu H = (D \kappa^4 + 1 - i) \kappa H \tanh(\kappa H)$$  \hspace{1cm} (16)

Having this in mind, it is expected that the roots $\{k_n, n = 0, 1, 2, \ldots\}$ are similarly distributed on the complex plane, as shown in Fig. 2. Only the one-half symmetric subset of the roots, shown in Fig. 2 by open circles, is needed in the expansion of equation (11).

The indexing of the roots of equation (16) is as follows: $\kappa_0$ is the real-positive root, $\kappa_1$ is the root inside the first quadrant of the complex plane (i.e. $\text{Re}(\kappa_1) > 0$, $\text{Im}(\kappa_1) > 0$), $\kappa_2$ is the conjugate-symmetric of $\kappa_1$ (thus, $\kappa_2 = -\text{Re}(\kappa_1) + i\text{Im}(\kappa_1)$), and $k_m$, $n = 3, 4, 5, \ldots$ are the roots lying on the positive-imaginary axis ($\text{Im}(k_m) > 0$). Thus, in the present expansion, the term $\varphi_0(x)Z_0(z)$ corresponds to the propagating mode, the terms $\varphi_1(x)Z_n(z)$, $n = 1, 2$, correspond to the decaying-propagating modes, and the remaining terms $\varphi_n(x)Z_n(z)$, $n = 3, 4, \ldots$, express the evanescent modes, that are especially important in the vicinity of ends $x_1$ and $x_2$ or places where the horizontal distribution of plate parameters present variations.

In contrast to the set of functions $Z_n(z; x)$, $n = 0, 1, 2, 3, \ldots$, which all satisfy $\partial Z_n(z = -h(x); x)/\partial z = 0$ on the bottom $z = -h(x)$, the term $Z_{-1}(z; x)$ is taken to be a smooth $z$-function satisfying the following inhomogeneous condition on the seabed

$$\frac{\partial Z_{-1}(z = -h(x); x)}{\partial z} = 1$$  \hspace{1cm} (17a)

Equation (17a), in conjunction with equation (11) after a termwise $z$-differentiation, leads to the following interpretation of the amplitude of the sloping-bottom mode

$$\varphi_{-1}(x) = \frac{\partial \varphi^{(2)}(x, z = -h(x))}{\partial z}$$  \hspace{1cm} (17b)

A consequence of equation (17b) is that the sloping-bottom term $\varphi_{-1}(x)Z_{-1}(z; x)$ identically vanishes on the horizontal parts ($h'(x) = 0$) of the bottom. A specific convenient form of $Z_{-1}(z; x)$ is given by

$$Z_{-1}(z; x) = h(x) \left[ \left( \frac{z}{h(x)} \right)^3 + \left( \frac{z}{h(x)} \right)^2 \right]$$  \hspace{1cm} (18)

and more details can be found in section 4 of Belibassakis and Athanassoulis [31].

3 ELASTIC PLATE MODELS

Plates and shells being real-world bodies can be dynamically analysed by means of the 3D elasticity equations or on any other appropriate material modelling (e.g. plasticity, elastoplasticity, viscoelasticity, etc.), in accordance with their dominating material behaviour. Apart from the geometrical feature of thinness, another important property which is always assumed as being inherent in any
thin body deserving to be characterized as a plate, is its ability to resist any type of deformation either stretching or bending.

In order to avoid special subtleties and pitfalls of general 3D numerical methods when applied to thin bodies, the specific geometrical feature of body thinness is explicitly used, leading to a mathematical remodelling facilitating better understanding of various specific features of body’s dynamic responses. In the following section a short review of such models for plates (or beams), is presented. It should be noted that the methodology can be also used to treat shells and other types of solid body with similar geometrical features.

### 3.1 Classical plate theory, first-order and third-order models

On the basis of general 3D elastodynamics, plate models are achieved by means of suitable expansions of the displacement field \( u_i(x,y,z;t) \), \( i = 1, 2, 3 \), with respect to the thickness direction \( z \in [-b/2, b/2] \), where \( z \) is usually measured (positive upwards) from the mean surface and \( b \) denotes the thickness. Turning for simplicity to the two-dimensional (2D) case (corresponding better to a beam than a plate), where all displacements take place in the \( xz \)-plane, a generic way to perform the reduction of dimensionality to only the horizontally dependent unknown fields, is by means of series expansions of the form

\[
 u_i(x, z; t) = \sum_{n=0}^{\infty} Z_n^{(i)}(z) u_n(x; t) \tag{19}
\]

where the \( Z_n^{(i)}(z) \) are selected to be some basis (or complete system) of functions defined in \( z \in [-b/2, b/2] \). To make the above representation efficient \( Z_n^{(i)}(z) \) needs to be selected so that fast convergence rates are achieved and the transverse (in the \( z \)-direction) infinite series can be restricted to finite sums with a small number of significant terms.

There are two different lines of thought that have been followed in introducing simplifying assumptions. The traditional approach, initiated by Poisson and Kirchhoff (see, e.g. Timoshenko and Woinowsky-Krieger [46]) gives priority to physical arguments and states the assumption in terms of physical variables such as strains and stresses, by requiring that the tractions on surfaces parallel to the middle surface are negligible, and strains vary linearly within the plate thickness. Although this approach is insightful and very efficient to start with, it has the disadvantage that it becomes very complicated if applied to better (higher-order) approximate models and also it neglects shear deformations.

The other approach is mainly based on analytical arguments concerning the form and nature of equation (19). In this context, on the basis of a Taylor expansion, a very reasonable model is obtained by using \( Z_n^{(i)}(z) = z^n \)

\[
u_i(x, z; t) = \sum_{n=0}^{N(i)} z^n u_n(x; t) \tag{20}
\]

where the number of terms retained, \( 1 + N(i) \), may be different for each \( i \)-component of the displacement. In all thin-plate theories the assumption of transverse inextensibility is considered to hold. Thus, in equation (20) the value of \( N(i) \) is taken to be zero, that is

\[
u_i(x, z; t) = w_0(x; t) \tag{21}
\]

Furthermore, by considering that the rest of the displacements have the same number of terms \( 1 + N \) the various thin-plate theories can be characterized in terms of their order \( N \). For example, first-order theories (e.g. Mindlin [47]) are based on the expansion

\[
u_i(x, z; t) = x_0(x; t) + z u_i(x; t),
\]

\[w(x, z; t) = w_0(x; t) \tag{22}\]

involving, in general, three horizontal surface fields (which in 3D plate bending becomes five). First-order models include the classical plate theory, which further assumes that the function \( u_i(x; t) \) (and \( u_0 \) in 3D plates) are expressed in terms of the deflection \( w_0(x,y;t) \) through

\[
u_i(x; t) = -\frac{\partial w_0(x; t)}{\partial x} \tag{23}
\]

A significant improvement of plate (and beam) theory is provided by third-order models (\( N = 3 \), involving in general five 2D fields or nine in 3D plate bending). Among the various third-order models appearing in the literature (reviewed by Wang et al. [37] and Reddy [48]), perhaps the most interesting one is that developed by Reddy [35] and Bickford [36], based on the following displacement expansion

\[
u_i(x, z; t) = u_0(x; t) + z u_1(x; t) + \left( -\frac{4}{3b^2} \right) z^2 \left( u_1 + \frac{\partial w_0}{\partial x} \right) \tag{24}\]
The above representation of the displacement field clearly allows for a quadratic variation of transverse shear strains and stresses, vanishing at the top and bottom surfaces of a plate, which is the correct boundary condition violated by all lower (than three)-order plate theories. Concerning the satisfaction of the latter condition, the Reddy–Bickford model, equations (24) and (25), is consistent only for plates with essentially horizontal boundaries. In the case of more general boundaries, such as the lower surface of the floating plate modelling the thick ice sheet in Fig. 1, an additional fourth-order term (or plate mode) is required for the consistent obtaining of a zero shear stress on the lower boundary \( z = -b(x) \). This term is needed only in the case of a sloping lower plate surface and for the case of a horizontal flat boundary it is zero. Based on equation (26) the non-zero components of the strain tensor are calculated as follows

\[
\varepsilon_{xx} = \frac{\partial u_x}{\partial x} = u_{0,x}(x) + z u_{0,x}(x) + \left( z^2 + \frac{2z^3}{3b} \right) u_2(x) + z^4 u_6(x)
\]

\[
\gamma_{xz} = 2 \varepsilon_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} = 2 \left( z + \frac{z^2}{b} \right) u_2(x) + 4z^3 u_6(x)
\]

and thus, \( \gamma_{xz} = 0 \) on \( z = 0 \). On the other hand, from equation (28) it can be seen that if the term \( z^4 u_6(x) \) is not included in the representation (26), then \( \gamma_{xz} = 0 \) on \( z = -b(x) \), which would be correct only in the case of flat surface \( b' = 0 \); c.f. equation (9). In the case of a general boundary \( z = -b(x) \), the plate mode \( u_6(x) \) is the additional degree of freedom of the present higher-order representation, allowing the

### 3.2 An enhanced vertical expansion of the displacement field in the floating plate

Taking into account the specific geometric characteristics of the thick plate with horizontal upper and general lower boundary shown in Fig. 3, the expansion of the complex displacement fields, in the present time-harmonic case, will be of the form

\[
u_{x}(x, z) = u_{0}(x) - zw_{0}(x) + \left( z^2 + \frac{2z^3}{3b} \right) u_2(x) + z^4 u_6(x),
\]

\[
u_{z}(x, z) = w(x)
\]
non-zero boundary $xz$-shear needed to satisfy the condition of vanishing tangential shear stresses, see equation (9), to be obtained.

This section is concluded by noting that in the case of a plate with a second variable upper boundary ($z = a(x)$, shown by dashed line in Fig. 2) one more additional mode is necessary. In this case, the expansion (26) for $u_a(x,z)$ is amended by an extra fifth-order term $z^5u_a(x)$. This remark can be exploited as starting point for developing a novel hierarchy of models for plates or beams with non-flat upper and lower boundaries.

4 THE CMS

By changing the term $(\partial\sigma_{ij}/\partial x)\delta u_i$ that appears in equation (7) back to $-\sigma_{ij}\delta u_i$, and using the result, in conjunction with the enhanced representations (26), (27), and (28), and the fact that in the present case, at $z = -b(x)$ it is possible to write

$$
\sigma_{xx} = \frac{E}{1 - \nu^2} e_{xx} + \frac{v}{1 - \nu} e_{zz}, \quad \text{and}
$$

$$
\sigma_{zz} = \mu e_{xx} = \frac{E v}{(1 + v)(1 - 2v)} e_{xx}
$$

We note here that in thin-plate theory the effect of $\sigma_{zz}$ stresses and their integral contribution in the plate section is neglected, which is recovered by the present model by setting $e_z = 1$.

4.1 Coupled equations concerning the elastic plate modes

Using the expansions (27) and (28) for $\epsilon_{xx}$ and $\gamma_{xx}$ and the corresponding ones for $\sigma_{xx}, \sigma_{zz}$ and $\sigma_{xz} = G_{y'zz}$, and calculating the expressions of $\delta u_i = (\partial u_i/\partial t)_{\text{eq}}$ in equation (31) in terms of the newly introduced elastic plate modes $\delta u_0, \delta u_2, \delta u_4$, and $\delta u$, it is possible to obtain from the equation associated with the variation $\delta w_0 = \delta w$ that

$$
-\omega^2 \rho_4 b w + \omega^2 \rho_T \frac{d}{dx} \left( \frac{b^2 u_0}{2} + \frac{b^2 u_2}{3} + \frac{7b^4 u_4}{60} + \frac{b^6 u_6}{6} \right) - M_{xx} x + (bB)_x + (-i\omega\rho_4)_{\text{wave}} = 0
$$

where $M$ is the plate bending moment

$$
M = \int_{z = 0}^{z = -b} z\sigma_{xx} \, dz
$$

$$
= \frac{E e_r}{1 - \nu^2} \int_{z = 0}^{z = -b} z^2 \epsilon_{xx} \, dz
$$

$$
= -\frac{E e_r}{1 - \nu^2} \left( \frac{b^3}{3} u_{xx}^2 + \frac{b^5}{2} u_{0,x} + \frac{7b^4}{60} u_{2,x} + \frac{b^6}{6} u_{0,x} \right)
$$

In deriving equations (29) and (30) it has been assumed that the following relations hold in the present case (see, e.g. section 8.3.1 in Graff [38]), using also equation (26)
and the term $B$ appearing in the left-hand side of equation (32a) comes from the contribution of shear stresses on $z = -b(x)$, equation (29), and is defined as follows

$$B = \frac{E\varepsilon_{zz}}{1-\nu^2}b_x \left( bu_x'' + u_0 + \frac{b^2}{3} u_x'' + b^4 u_0'' \right) - 4Gb^2 u_{b,x}$$

(32c)

Similarly, it is possible to obtain the following equations from the terms of the variational principle (31) associated with the variations $\delta u_0$, $\delta u_x$, $\delta u_b$, respectively

$$-\omega^2 \rho_E b \left( u_0 + \frac{b}{2} u_x + \frac{b^2}{6} u_x'' + \frac{b^4}{5} u_b \right) - E\varepsilon_{zz} \frac{\partial}{\partial \xi} \left[ \left( bu_0 + \frac{b^2}{2} u_x'' + \frac{b^3}{6} u_x'' + \frac{b^5}{5} u_0'' \right) \right] - B = 0$$

(33)

$$-\omega^2 \rho_E b^3 \left( \frac{u_0}{6} + \frac{7b}{60} u_0'' + 0.0413 b^5 u_2 + 0.0595 b^b u_b \right) - \frac{Bb^2}{3} + 2Gb^3 \left( 0.0667 u_2 + 0.133b^2 u_{b,x} \right) - E\varepsilon_{zz} \frac{\partial}{\partial \xi} \left[ \left( \frac{b^2}{3} u_0'' + \frac{7b^3}{60} u_0'' + 0.0413b^5 u_2 + 0.0595b^b u_{b,x} \right) \right] = 0$$

(34)

$$-\omega^2 \rho_E b^5 \left( \frac{u_0}{5} + \frac{b}{6} u_x'' + 0.0595b^2 u_2 + \frac{b^4}{9} u_b \right) - Bb^4 + 4Gb^5 \left( 0.0667 u_2 + 0.5714b^4 u_b \right) - E\varepsilon_{zz} \frac{\partial}{\partial \xi} \left[ \left( \frac{b^2}{5} u_0'' + \frac{b^6}{6} u_0'' + 0.0595b^2 u_2 + 0.0595b^b u_{b,x} \right) \right] = 0$$

(35)

Equations (33) to (35) constitute a system of horizontal differential equations with respect to the elastic plate modes $w(x), u_0(x), u_x(x)$, and $u_b(x)$, that is furthermore coupled with the wave potential through its values on the lower plate boundary $\varphi(z = -b) = \varphi(x, z = -b(x))$ appearing in equation (32a). We note here that in the case of a flat lower plate boundary ($b'_x = 0$ and $u_0 = 0$) then from equation (32c) we obtain that $B = 0$, and thus equations (33) to (35) can be simplified by dropping the corresponding terms involving $B$ and $u_{b,x}$.

An important remark concerning the above system and its purely elastic version, which is obtained by dropping the last term in the left-hand side of equation (32a) (i.e. the forcing of the elastic plate by the hydrodynamic pressure acting on its lower surface in the context of linearized water wave theory), is that it actually constitutes a novel, enhanced model for shear deformable plates and beams of general thickness that could find useful applications in the study of plate/beam vibrations in vacuo. This is left to be presented in more detail in a separate work.

### 4.2 Coupled equations concerning the modes of the wave potential

Using the local-mode expansion of the wave potential $\varphi(x,z) = \sum_n \varphi_n(x) Z_n(z;x)$ in the water region and the corresponding expression concerning the variations $\delta \varphi$ in terms of the hydroelastic wave modes $\delta \varphi_n$ in the variational principle, the following CMS of horizontal differential equations for $\varphi_n(x)$ is obtained (from the last three terms in the left-hand side of equation (31))

$$\sum_{n=1}^{\infty} \alpha_{mn}(x) \frac{\partial^2 \varphi_n(x)}{\partial x^2} + b_{mn}(x) \frac{\partial \varphi_n(x)}{\partial x} + c_{mn}(x) \varphi_n(x) = i \omega w(x), \quad m = -1, 0, 1, \ldots$$

(36)

The $x$-dependent coefficients of the above CMS are defined as follows

$$\alpha_{mn}(x) = \left< Z_n, Z_m \right>_h$$

(37a)

$$b_{mn}(x) = 2 \left< \frac{\partial Z_n}{\partial x}, Z_m \right>_h$$

(37b)

$$c_{mn}(x) = \left< \frac{\partial^2 Z_n}{\partial x^2} + \frac{\partial Z_n}{\partial x}, Z_m \right>_h$$

(37c)
where \( \langle f, g \rangle_{-b} = \int_{-b}^{z=-b(x)} f(z) g(z) dz \). The expansion of the wave potential, equation (11), can now be used to calculate its values on the lower plate surface, \( \varphi|_{z=-b} = \varphi(x, z = -b(x)) = \sum_{n=0}^{\infty} \varphi_n(x) \), and use this expression in equation (32a) which takes the form

\[
\mathcal{A}_t(w, u_0, u_2, u_b) = \frac{i \mu}{\omega} \sum_{n=1}^{\infty} \varphi_n(x) \tag{38}
\]

where \( \mathcal{A}_t \) is a differential operator on the horizontal fields \( w, u_0, u_2, u_b \) defined by

\[
\mathcal{A}_t(w, u_0, u_2, u_b) = \left(1 - \frac{\omega^2 \rho b}{\rho g} \right) w + \frac{\omega^2 \rho_b}{\rho g} \frac{\partial}{\partial x} \left(b^2 u_0 + \frac{b^3 w}{2} + \frac{7b^4 u_2}{60} + \frac{b^5 u_b}{6}\right) + \frac{M_{xx}(b b \varphi)}{\rho g} \tag{39}
\]

Thus, the present hydroelastic problem is finally reduced to the CMS (36) which is further coupled with the system governing the elastic plate modes through equation (38), in conjunction with equations (33), (34), and (36) bringing the effects of the rest of the plate modes in the present higher-order model.

### 4.3 Periodic solution in homogeneous environment. Enhanced dispersion relation

In areas where the depth and the plate thickness are constant, the CMS (36) is greatly simplified. First, all terms associated with \( \varphi_{-1} \) can be dropped, since \( \varphi_{-1} = 0 \), when \( h'(x) = 0 \) (c.f. equation (17b)). In addition, the coefficients of the CMS become constant, as obtained by dropping the terms containing \( x \)-derivatives in equations (37a) to (37c). Then, the present CMS takes the form

\[
\sum_{n=0}^{\infty} \langle Z_n, Z_m \rangle (\varphi''_n(x) + \kappa_n^2 \varphi_n(x)) - f_n \varphi_n(x) = i \omega w(x), \quad m = 0, 1, 2, 3, \ldots \tag{40a}
\]

where from equation (14)

\[
f_n = \kappa_n \tanh(\kappa_n h) \tag{40b}
\]

and

\[
\mathcal{A}_t(w, u_0, u_2, u_b) = \frac{i \mu}{\omega} \sum_{n=0}^{\infty} \varphi_n(x) \tag{40c}
\]

Furthermore, if the lower plate surface is flat horizontal, and its thickness constant \( b(x) = b \), then \( u_0 = 0 \) and the terms involving \( B \) can be dropped. Seeking for spatially periodic solutions of the form \( \varphi_n, w, u_0, u_2 \sim \exp(\pm i k x) \), it can be easily obtained from equations (33), (34), and (35) that the elimination relations of \( u_0 \) and \( u_2 \) in terms of \( w \) are

\[
u_0 = \left(1 - \frac{0.0055 b^2}{0.0135 b^2 + X}\right)(-i k b w) \tag{41a}
\]

\[
u_2 = \left(\frac{0.0332}{0.0135 b^2 + X}\right)(-i k b w) \tag{41b}
\]

where the newly introduced parameter \( X/b^2 \) is defined as follows

\[
\frac{X}{b^2} = \frac{0.132(G/E) e_i^{-1}}{\kappa^2 b^2 \left[1 - (1/12)(e/\varepsilon)(\kappa h)^{-2}\right]} = \frac{0.132(1 - \varepsilon)}{2 \kappa^2 b^2 e_i \left[1 - (1/12)(e/\varepsilon)(\kappa h)^{-2}\right]} \tag{42}
\]

The above parameter defined by equation (42) carries the effect of shear modulus \( G/E = 1/2(1 + \nu) \) and \( e_i \neq 1 \). Also, apart from the shallowness parameter \( \kappa h \) and the plate relative thickness \( b/H \), it involves the plate mass and rigidity parameters

\[
\varepsilon = \frac{\rho \varepsilon_b \varepsilon_o b^2}{\rho g}, \quad D = \frac{D_e}{H^4} = \frac{E b^3}{12 \rho g (1 - \nu^2)} H^4 \tag{43}
\]

the latter being expressed in terms of the classic plate flexural rigidity coefficient

\[
D = Eb^3 \left/ \left[12 \rho g (1 - \nu^2)\right]\right. \tag{44}
\]

Using the above results in equation (40c) it is possible to obtain for the case of a homogeneous floating plate in a constant depth strip the following equation

\[
\left[D \kappa^4 H^4 (1 - \delta) + 1 - \varepsilon \left(1 + \frac{\kappa^2 b^2}{12}(1 - \delta)\right)\right] w = \frac{i \mu}{\omega} \sum_{n=0}^{\infty} \varphi_n(x) \tag{45}
\]

where, the parameter \( \delta \) is defined as follows.
\[ \delta = \frac{0.0135}{0.0135 + (X/b^2)} \]  

(46)

Also, expressing the general solution of the CMS (40a) in the form

\[ \phi_n(x) = \chi_n \exp(i\kappa_n x) + \beta_n \exp(-i\kappa_n x), \]

\[ n = 0, 1, 2, 3, \ldots \]  

(47)

where \( \kappa_0 \) stands for the propagating mode and \( \kappa_n \), \( n = 1, 2, \ldots \) for the evanescent modes, in accordance with the discussion in section 2.2 (see also Fig. 2).

The latter, in the case of periodic waves in the infinite strip considered, disappear \( \chi_n = \beta_n = 0 \), \( n = 1, 2, \ldots \). Consequently, substituting equation (47) into equation (40a) it is possible to obtain

\[ w(x) = \frac{i}{\alpha} \sum_{n=0}^{\infty} f_n \phi_n(x) = \frac{i}{\alpha} \sum_{n=0}^{\infty} \kappa_n \tanh(\kappa_n h) \phi_n(x) \]

(48)

Substituting the above result into equation (46) it is clear that the latter is satisfied if \( \{ \kappa_n, n = 0, 1, 2, 3, \ldots \} \) are obtained as the roots of the following hydroelastic dispersion relation (in non-dimensional form)

\[ \left\{ D\kappa^4b^4(b/H)^{-4}(1-\delta) + 1 - \varepsilon \left[ 1 + \frac{\kappa^2b^2}{12}(1-\delta) \right] \right\} \times \kappa H \tanh(\kappa H) = \mu H \]

(49)

where the non-dimensional parameters \( \varepsilon \), \( D \), and \( \delta \) have been defined by equations (43) and (46), respectively. The above result justifies our selection for the function \( \chi(\kappa) \) concerning the formulation of the vertical eigenvalue problem

\[ \chi(\kappa) = D\varepsilon, \kappa^4(1-\delta) + 1 - \varepsilon \left[ 1 + \frac{\kappa^2b^2}{12}(1-\delta) \right] \]

(50)

In the case of plates of small thickness \( b/H \rightarrow 0 \), and thus \( \kappa b \rightarrow 0 \), then from equations (42) and (46) it can be shown that the parameters \( X/b^2 \rightarrow \infty \) and \( \delta \rightarrow 0 \). If, in addition, the effects of vertical stresses \( (\varepsilon, \varepsilon) \) are disregarded the present hydroelastic dispersion relation reduces to its form obtained by classical thin-plate theory, that is

\[ \chi(\kappa) = D\kappa^4 + 1 - \varepsilon \]

(51)

The above asymptotic form is also exploited in developing the algorithm for calculating the roots \( \kappa_n \), \( n = 0, 1, 2, 3, \ldots \) of equation (49) on the complex plane (see Fig. 2), in the general case. First, the roots are approximated by using the thin-plate hydroelastic dispersion relation, equation (16). On the basis of the starting approximation, the values of the parameters \( \kappa b, X/b^2 \), and \( \delta \) involved in equation (49) are calculated, and the procedure continues with iterations until convergence. Usually, for non-exotic values of the material parameters and moderate plate thickness, two or three iterations are enough for convergence in the general case.

The effects of the various parameters on the dispersion relation are illustrated in Figs 4 and 5, that present results obtained from equation (49) concerning the phase speed of waves \( C = \omega/\kappa \), with \( \kappa = \kappa_0 \), in terms of the non-dimensional wavenumber \( \kappa H \), which is inversely proportional to the shallowness parameter \( (\kappa H = 2\pi H/\lambda) \). For reasons of comparison, first the results obtained from the classical thin-plate theory, equation (16), without any shear effects on the plate deformation, and using \( \varepsilon = 1 \) are presented in Fig. 4. More specifically, the phase speed of propagating hydroelastic waves, non-dimensionalized with respect to the phase speed of linearized water waves in shallow conditions \( (C/\sqrt{gH}) \), are plotted for three representative values of \( D = 1, 10, \) and 100 and three values of \( \varepsilon = 0, 0.5, \) and 1, and for \( \kappa H \) ranging from very shallow to very deep water conditions \( 0 < \kappa H < 16 \). A density ratio \( \rho_f/\rho = 923/1025 \) corresponding to ice/water has been used. Noting that the vertical axis is log-scaled, it can be clearly observed in Fig. 4 that the effect of flexural rigidity is that it leads to a significant increase in the phase speed. Also, in the limit of very shallow conditions \( \kappa H \ll 1 \), it is possible to observe from this figure the effect of the mass parameter \( \varepsilon \), for each considered value of the flexural rigidity. Clearly, for \( \kappa H \rightarrow 0 \), equation (16) gives that \( C/\sqrt{gH} \approx 1 - \varepsilon \), and this explains the bend of the curves (for each \( D \)) near zero (as shown in the included subplot of Fig. 4).

Figure 5 presents corresponding results obtained from the enhanced dispersion relation equation (49), for increased plate thickness \( bh = 0.5 \) and Poisson ratio \( \nu = 0.3 \). In this case, the shear parameter \( G/E = 1/2(1+\nu) = 0.38 \), which is close to values for ice. As for the effect of Poisson ratio, it is noted that, for small and moderate values of thickness, no important difference is obtained by changing \( \nu \) from 0.05 to 0.5, corresponding to the range of Poisson ratio for non-auxetic materials. On the other hand, a drop in phase speed of the hydroelastic waves, as
Fig. 4  Phase speed of hydroelastic waves for various values of the shallowness parameter $\kappa H$, ranging from very shallow to very deep conditions, based on classical thin-plate theory equation (16). Results are shown for three values of $D = 1, 10, \text{ and } 100$ and three values of $\varepsilon = 0, 0.5, \text{ and } 1$ (see equation (43)).

Fig. 5  Same as in Fig. 4, but now based on enhanced dispersion relation equation (49), for a finite and uniform plate thickness $b/h = 0.5$ and a Poisson ratio $\nu = 0.3$. 

compared with the results for thinner ice can be observed in Fig. 5. For example, at $\kappa H = 2$, it can be seen that, for all values of the flexural rigidity parameter, the phase speed of the hydroelastic waves for a thickness ratio $b/h = 0.5$ is about 67 per cent lower than the corresponding value for $b/h = 0.1$, which is very close to the one obtained on the basis of thin-plate theory. This result shows that, for the case of relatively thick ice, the hydroelastic dispersion effects may be poorly modelled by thin-plate models.

4.4 General representations in the semi-infinite constant-depth strips

To treat the present hydroelastic problem in the case of infinite domains (Fig. 1), the water plus elastic plate domain $D = D_0 \cup D_E$ is decomposed in three subdomains $D^{(i)}$, $i = 1, 2, 3$, defined as follows: $D^{(1)}$ and $D^{(3)}$ are homogeneous constant-depth subdomains, characterized by $x < x_1$ and $x > x_3$, respectively, and $D^{(2)}$ is the variable bathymetry subdomain, lying between $D^{(1)}$ and $D^{(3)}$, containing also the floating elastic plate irregularity. Finally, the vertical interfaces separating the three subdomains are defined as the vertical segments (between the bottom and the mean water level) at $x = x_1$ and $x = x_3$, respectively, shown by dashed lines in Fig. 1. Noting that in the two half-strips the depth is constant, the sloping-bottom mode is zero. Based on the analysis presented in section 4.4, the general representations in the two semi-infinite strips $D^{(1)}$ and $D^{(3)}$ are derived as follows

$$\varphi^{(1)}(x, z) = \left[ \exp(ik^{(1)}_0 x) + A_R \exp(-ik^{(1)}_0 x) \right] Z^{(1)}_0(z) + \sum_{n=1}^{\infty} C^{(1)}_n Z^{(1)}_n(z) \exp(-ik^{(1)}_n(x-x_1)),$$

in $D^{(1)}$ \hspace{1cm} (52)

$$\varphi^{(3)}(x, z) = A_T \exp(ik^{(3)}_0 x) Z^{(3)}_0(z) + \sum_{n=1}^{\infty} C^{(3)}_n Z^{(3)}_n(z) \exp(-ik^{(3)}_n(x_3-x)),$$

in $D^{(3)}$ \hspace{1cm} (53)

where $\kappa^{(i)}_n$, $Z^{(i)}_n(z)$, $i = 1, 3$, are the eigenvalues (obtained as the roots of equation (49)) and eigenfunctions, respectively, of the problem equations (12) and (13) formulated at constant depth $h_i$ and plate thickness $b_i$, characterizing the domains $D^{(i)}$, $i = 1, 3$. Moreover, $A_R$ and $A_T$ denote the reflection and transmission coefficients, respectively, and $C^{(i)}_n$, $i = 1, 3$ are the coefficients of the rest of the (evanescent) modes in each half strip.

Using the above expansions the problem is finally reformulated as a transmission boundary value problem governed by the present CMS, equations (36) and (38), in conjunction with equations (33), (34), and (35) for the rest of the elastic plate modes in the middle domain $D^{(2)}(x_1 < x < x_3)$, where both plate parameters and the bathymetry may present variability. In this case, the following conditions are derived by matching of the wave field at the vertical interface separating $D^{(1)}$ and $D^{(2)}$

$$\varphi^{(1)}_0(x_1) + ik^{(1)}_0 \varphi^{(1)}_0(x_1) = 2i \kappa^{(1)}_0 \exp(ik^{(1)}_0 x_1),$$

$$\varphi^{(2)}_n(x_1) + ik^{(1)}_n \varphi^{(1)}_n(x_1) = 0, \hspace{1cm} n = 1, 2, 3, \ldots \hspace{1cm} (54)$$

and similar ones can be obtained from the matching of the wave field at the vertical interface separating the subdomains $D^{(2)}$ and $D^{(3)}$. Additional conditions are obtained from the form of the present mode system in the homogeneous strip, equations (40a) to (40c). The latter conditions, if the ends $(x_1, x_3)$ are taken a distance of the order of one to two wavelengths away from the inhomogeneity, reduce approximately to (see, also, equations (47) and (48))

$$\varphi^{(2)}_n(x_1) + \left( \kappa^{(1)}_n \right)^2 \varphi^{(1)}_n(x_1) = 0, \hspace{1cm} at \hspace{0.2cm} x = x_1 \hspace{1cm} (55)$$

and similarly at $x = x_3$. The end-conditions of equations (54) and (55) at each end $x_i$, $i = 1, 3$, are sufficient for the integration of the present, fourth-order, CMS in $D^{(2)}$.

4.5 Numerical examples and discussion

The discrete version of the present CMS, in the case of an infinite raft, is obtained by truncating the local-mode series of equation (11) to a finite number of terms (modes), and using central, second-order finite differences to approximate the horizontal derivatives. Discrete boundary conditions are obtained by using second-order forward and backward differences to approximate the horizontal derivatives in equations (54) and (55) and the corresponding ones at $x = x_3$. Thus, the discrete scheme is uniformly of second order in the horizontal direction. The coefficient matrix of the discrete system is block structured. The system matrix has a total dimension $(N_m + 2) (M + 1)$, where $N_m$ denotes the index where the series of equation (11) is truncated and $M$ is the
number of segments subdividing the interval \( x_1 \leq x \leq x_2 \).

A specific example is presented in Fig. 6 concerning the propagation of harmonic hydroelastic waves of period \( T = 12 \) s (\( \omega = 0.523 \) rad/s) on a floating ice sheet, characterized by a modulus of elasticity \( E = 5 \) GPa, a Poisson’s ratio \( \nu = 0.3 \), and values of ice/water densities 923/1025 kg/m\(^3\) (see, also, Squire et al. [2]). For this case, except for a uniform ice sheet of finite thickness \( b = 1 \) m, at a constant depth \( h = 10 \) m, also considered is the effect of inhomogeneous ice thickness for \( 100 < x < 400 \) m, with a specific form as given by equation 5.22 in Porter and Porter [1]. In this case the ice thickness varies from 1 m at the ends of the domain to 3 m in the middle part, as shown in the lower part of Fig. 6. Also considered is the effect of a shoal, extending from \( x = 0 \) m to 500 m, connecting two regions of constant but different depths: the left region of wave incidence, where \( h_1 = 13 \) m, and the region of transmission (right half strip), where \( h_3 = 7 \) m. In the latter case, the average and maximum values of the slope of the bottom profile are 1.2 per cent and 6 per cent, respectively.

The horizontal distribution of the modulus of vertical sheet deflection (\( w \)) normalized with respect to the incident wave amplitude is shown in the upper part of Fig. 6. It can be clearly observed in this figure that the effect of a shoal with a small sloping bottom on the elastic body motion is an increase of the deflection of the order of 15 per cent, in comparison with a flat domain of the same average depth. Also, the effect of horizontal variability of ice sheet thickness on the deflection can be seen to be an increase of the maximum deflection of the order of 3 per cent which occurs around the middle of the domain.

Finally, in Fig. 7 the moduli of the modal-amplitude functions, i.e. the quantities \( |\varphi_n(x)| \) in \( x_1 \leq x \leq x_2 \), are plotted, as obtained by the presented method for the previous example, in the case of an inhomogeneous ice sheet above the shoal. The horizontal axis of Fig. 7 is a multiple replica of the interval \([x_1,x_2]\), i.e. a sequence of repeated intervals each one associated with a mode and named after the mode number. In the \( m\)th replica of \([x_1,x_2]\) the amplitudes \( |\varphi_n(x)| \) of the \( n\)th mode, are plotted, using solid lines, respectively. It was observed from this figure (and many similar results keeping more terms in the expansion) that the decay of the modal amplitudes is of the order \( O(n^{-4}) \), which is sufficient to ensure the uniform convergence (up to and including the boundaries) of the local-mode series and its derivatives. This fact also guarantees the fast algebraic convergence of the specific representation, allowing accurate results to be obtained by keeping a small number of terms (e.g. 5 to 6) in the local-mode expansion (11).

5 A SLOWLY VARYING APPROXIMATION CONCERNING PLATE THICKNESS

For the case of plates of general thickness, but with a slowly varying lower (wetted side), it would be
expected that the elimination relations given by equations (41a) and (41b), which are valid for homogeneous plates, are approximately correct and $u_b = 0$. The analysis presented in section 4.3 allows a simplified form of the present time-harmonic, coupled-mode hydroelastic system, to be obtained for floating shear deformable plates of slowly varying thickness and mass distributions, but in general bathymetry regions. In this case, the system can be described by equation (36)

$$\sum_{n=1}^{\infty} a_{mn}(x) \frac{\partial^2 \varphi_n(x)}{\partial x^2} + b_{mn}(x) \frac{\partial \varphi_n}{\partial x} + c_{mn}(x) \varphi_n(x) = i\omega w(x),$$

$$m = -1, 0, 1, \ldots$$

(56)

where the $x$-dependent coefficients $a_{mn}(x)$, $b_{mn}(x)$, and $c_{mn}(x)$ of the above CMS are given by equations (37a), (37b), and (37c), in conjunction with the following equation

$$\frac{D^2}{Dx^2} \left[ DH^4 (1-\delta) \frac{\partial^2 w}{\partial x^2} \right] + \epsilon (1-\delta) \frac{b^2}{12} \frac{\partial^2 w}{\partial x^2} + (1-\epsilon) w = i\mu \sum_{n=0}^{\infty} \varphi_n(x)$$

(57)

where $\epsilon$ and $D$ were defined in equation (43) and equation (44), respectively. Equation (57) is obtained from equation (39) using $B' = 0$ and the following approximate expressions concerning the rest of the plate modes in terms of the solution $w(x)$ of the above system

$$u_0(x) = \left( \frac{1}{2} - \frac{0.0055b^2}{0.0135b^2 + X} \right) (-i\kappa b w)$$

(58a)

$$u_2(x) = \left( \frac{0.0332}{0.0135b^2 + X} \right) (-1\kappa b w)$$

(58b)

$$u_b = 0$$

(58c)

where all the involved parameters are now considered to be $x$-dependent functions, and $\kappa = \kappa_0(x)$ is the local propagating wavenumber, corresponding to the $n = 0$ root of the generalized dispersion relation, equation (49), formulated at the local depth and using the local plate thickness. In deriving equation (57) the variability of the plate parameters has been assumed to be low, so that all $b'$, $X'$, and $\kappa_0$ are small, and thus it is possible to use

$$u_0(x) \approx \left( \frac{u_0}{w} \right) w'(x), \quad \text{and} \quad u_2(x) \approx \left( \frac{u_2}{w} \right) w'(x)$$

(59)

and similar relations approximating higher-order horizontal derivatives involved in equation (39). After solving the simplified version of the CMS of equations (56) and (57), the wave characteristics can be obtained over all the domain by means of the calculated wave modes $\varphi_n(x)$, $n = -1, 0, 1, 2, 3, \ldots$ using equation (11). Also, the elastic strain and stress distributions in the thick plate cross-section can be obtained from the solution, using equations (58a), (58b), and (58c), in conjunction with equations (27)
and (28), and the formulae

$$\sigma_{xx} = \frac{E\epsilon_{xx}}{1-\nu^2}, \quad \sigma_{zz} = \frac{Ev}{(1+\nu)(1-2\nu)}\epsilon_{zz}, \quad \sigma_{xz} = G\gamma_{xz}$$

(60)

Furthermore, in the case of plates of small thickness $b/H \to 0$, then, as already discussed in section 4.3, the parameters $X/b^2 \to \infty$ and $\delta \to 0$. If, in addition, the effects of vertical stresses in the sectional bending moment ($e_n R_0$) are disregarded, then equation (57) reduces to its thin-plate theory analogue, that is

$$L_2 \frac{L^2}{C_{18}/C_{19}}$$

In this case, the approximation of the CMS is defined by equations (56) and (61), where the $x$-dependent coefficients $a_{mn}(x)$, $b_{mn}(x)$, and $c_{mn}(x)$ of the CMS are obtained by equations (37a), (37b), and (37c), and are simplified as follows

$$a_{mn}(x) = \langle Z_n Z_m \rangle_{-h}$$
$$b_{mn}(x) = 2\langle Z_n Z_m \rangle_{-h} \frac{dh}{dx} Z_n(z = -h; x) Z_m(z = -h; x)$$
$$c_{mn}(x) = \langle \frac{\partial^2 Z_n}{\partial x^2} + \frac{\partial^2 Z_n}{\partial z^2} Z_m \rangle_{-h} \frac{\partial Z_n}{\partial z} (z = -h; x) + \frac{dh}{dx} \frac{\partial Z_n}{\partial x} (z = -h; x)$$
$$\times Z_m(z = -h; x)$$

(62a, 62b, 62c)

with $Z_n(z; x)$ being calculated from equation (14) using $H = h$, in conjunction with the local hydroelastic dispersion relation equation (15b) based on thin-plate theory (i.e. by using $\zeta = \omega_n(k) = (Dk^4 + 1 - \epsilon))$. The above version is the direct generalization of the hydroelastic CMS presented by Belibassakis and Athanassoulis [31] for the case of a floating plate of variable thickness, flexural rigidity, and mass parameters, in the context of thin-plate theory.

Figure 8 is a comparison of the modulus of the vertical deflection ($w$), between the results obtained by the presented CMS (shown by a solid line) and the ones from the model of equations (56) and (61) (shown by dashed line), for the case of an inhomogeneous ice sheet lying above the shoal of Fig. 6. It should be remembered that the approximate model does not take into account the effects of shear ($G/E$ and $\epsilon_n \neq 1$). In this example of relatively low values of ice sheet and bottom steepness ($|b' h'|(h') \ll 1$), the differences of the calculated deflection are maximized at the shallow end of the domain, and remain at the order of 1 to 2 per cent.

More details concerning the application of the present slowly varying approximation given by equations (56) and (61), in comparison with the corresponding simplified model in the context of thin-plate theory, equations (56), (61), (62a), (62b), and (62c), including examples of application to various interesting cases in variable bathymetry will be presented in Belibassakis and Athanassoulis [49].

6 CONCLUSIONS

In this work a novel CMS of horizontal differential equations has been derived for the hydroelastic
analysis of large floating bodies or ice sheets of general, finite thickness, lying over variable bathymetry regions. The present method is based on the theory of shear deformable plates (or beams), and is derived by an enhanced representation of the elastic displacement field, containing additional elastic vertical modes and permitting the shear strain and stress to vanish on both the upper and lower boundaries of the thick floating plate. This model extends third-order plate theories to plates and beams of general shape. The presented CMS is obtained by means of a variational principle composed of the one-field functional of elastodynamics in the plate region, in conjunction with the pressure functional in the water region. The wave potential in the water column is represented by means of a local-mode series expansion, containing an additional mode and providing an appropriate correction term on the bottom boundary, when the slope is not mild.

The theoretical value and practical effectiveness of the present model, apart from bringing the effects of finite thickness and shear into play, is that a small number of modes is enough for numerical convergence, even in cases of a general plate surface and for arbitrary depths. Extensive numerical evidence suggests that the rate of decay of the mode amplitudes is very fast and thus, truncation of the modal series to its first few terms (keeping five or six modes) is sufficient for an accurate solution. Numerical examples are presented, showing that useful results can be obtained for the hydroelastic analysis of thick, floating elastic bodies in inhomogeneous environments. Finally, it is worth noting that the analytical structure of the present model facilitates its extension to fully 3D problems, as well as to finite, heterogeneous raft configurations in water wave–ice interaction problems.

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