A coupled-mode model for water wave scattering by horizontal, non-homogeneous current in general bottom topography

K.A. Belibassakis*, Th.P. Gerostathisb, G.A. Athanassoulias

School of Naval Architecture and Marine Engineering, National Technical University of Athens, Zografos 15773, Athens, Greece
Dept. of Naval Architecture, School of Technological Applications, Technological Educational Institute of Athens, Ag. Spyridonos 12210, Athens, Greece

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ABSTRACT
A coupled-mode model is developed for treating the wave–current–seabed interaction problem, with application to wave scattering by non-homogeneous, steady current over general bottom topography. The vertical distribution of the scattered wave potential is represented by a series of local vertical modes containing the propagating mode and all evanescent modes, plus additional terms accounting for the satisfaction of the free-surface and bottom boundary conditions. Using the above representation, in conjunction with unconstrained variational principle, an improved coupled system of differential equations on the horizontal plane, with respect to the modal amplitudes, is derived. In the case of small-amplitude waves, a linearised version of the above coupled-mode system is obtained, generalizing previous results by Athanassoulias and Belibassakis [Fluid Mech 1999; 389:275–301] for the propagation of small-amplitude water waves over variable bathymetry regions. Keeping only the propagating mode in the vertical expansion of the wave potential, the present system reduces to an one-equation model, that is shown to be compatible with mild-slope model concerning wave–current interaction over slowly varying topography, and in the case of no current it exactly reduces to the modified mild-slope equation. The present coupled-mode system is discretized on the horizontal plane by using second-order finite differences and numerically solved by iterations. Results are presented for various representative test cases demonstrating the usefulness of the model, as well as the importance of the first evanescent modes and the additional sloping-bottom mode when the bottom slope is not negligible. The analytical structure of the present model facilitates its extension to fully non-linear waves, and to wave scattering by currents with more general structure.

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1. Introduction

The evolution of water-waves in nearshore and coastal areas, and especially, in regions where ambient, tidal and other type of steady-like currents exist, is important for a variety of engineering applications including interaction of waves with structures, coastal management, harbour maintenance, exploitation of renewable energy sources, etc. Sometimes the effects of currents on wave transformation appear significant, since they are responsible for Doppler shifting and additional wave refraction, reflection, and breaking, completely changing the wave energy pattern. In particular, the characteristics of surface waves present significant variation as they propagate through non-homogeneous ambient currents, in the presence of depth inhomogeneities in variable bathymetry regions. Thus, large amplitude waves can be produced when obliquely propagating waves interact with opposing currents; see, e.g., [1]. This phenomenon could be further enhanced by inshore effects due to sloping seabeds, and has been reported to be connected with the appearance of “giant waves” [2]. Extensive reviews on the subject of wave–current interaction in the nearshore region have been presented by various authors [3–5]. Recent information can also be found in the corresponding sections of reviewing articles [6,7].

The study of spatial evolution of water waves and the investigation of scattering of realistic wave spectra over irregular currents with characteristic variation lengths comparable to the dominant wavelength, including the effects of bottom irregularities, can be supported by theoretical models treating the simpler problem of monochromatic waves interacting with steady inhomogeneous currents. Wave–current interaction models over slowly varying bottom topography have been developed and studied by various authors. Under the assumption of irrotational wave motion, Kirby [8] derived a phase-resolving, one-equation model, generalizing Berkhoff’s [9] mild-slope equation in regions with slowly varying depth and currents; see also [10]. The latter model in its elliptic time-harmonic and parabolic forms has been exploited, in conjunc-
tion with numerical (finite-element, finite difference, etc.) solvers, to numerous wave–current–seabed interaction applications; see, e.g., [11,12] respectively, and the references therein. On the other hand, if the wave flow is assumed to be weakly rotational, as happens to be the case when waves are scattered by shearing currents, McKee [13,14] derived another one-equation model called the mild-shear equation. The validity of the mild-shear equation is also based on the assumption of slow current and depth variations compared to the typical wavelength. In the case of flat bottom, the mild-shear model has been further enhanced by McKee [15] by including an extra term and obtaining the enhanced mild-shear equation. The latter model is applicable to cases where the shearing current is varying on the scale of the wavelength. In the above works the current is considered to be flowing along one horizontal direction while the bottom topography varies in the other horizontal direction. Thus, the mild-shear model is more appropriate for problems of wave scattering by slowly varying depth and longshore-type ambient shearing currents.

In both the above approaches (mild-slope model, mild-shear model) the effects of evanescent modes, describing higher-order localised effects due to bottom and current variations, have been ignored. Based on complete normal-mode type expansions, another class of wave–current–seabed interaction models has been developed, applicable to cases where the lateral length scale on which the medium is changing is much smaller than the typical wavelength. In this case, the problem is treated by means of step discontinuities and vertical vortex sheets, separating subregions of essentially potential flow, in conjunction with appropriate matching conditions ensuring continuity of pressure and normal flow following the vortex sheet(s). In this context, generalising the work by Evans [16] for the transmission of deep-water waves across a vortex sheet, Smith [17,18] presented models for waves crossing uniform current jets in constant finite depth and crossing a step with horizontal shear, respectively. Also, Kirby et al. [19] studied the propagation of obliquely incident waves over a trench with uniform current flowing along it. In the latter works complete representations of the wave potentials in the various subregions have been used, containing both the propagating and the evanescent modes, which are necessary in order to satisfy the matching/boundary conditions at the vertical interfaces (vortex sheets and depth discontinuities).

The approach by Smith [18] and Kirby et al. [19] has been further exploited by McKee [20] to study scattering of waves by shearing currents of general horizontal structure in water of constant depth. In the latter work, a step-like approximation of the bathymetry is used and the current is modelled by a series of vertical vortex sheets separating subregions of constant current velocity. The solution is obtained by using complete representations of the wave potential in each subregion and matching conditions at the vertical interfaces. Finally, a continuous multi-mode model has been developed by Belibassakis [21] for the scattering of water waves by horizontally shearing currents in variable bathymetry regions, without asymptotic assumptions or restrictions concerning the bottom and current variation lengths with respect to the local wavelength. Keeping only the propagating mode, the above coupled-mode system simplifies to a new one-equation model called the mild-slope and shear equation, having the property to reduce to the modified mild-slope equation in the case of no-current, and to the enhanced mild-shear equation in the case of flat bottom.

In the present work we consider the scattering of waves by a non-homogeneous current of general horizontal structure in a variable bathymetry region, in the case where gravity forces are dominant, and thus, we can approximately ignore the effects of capillarity and viscosity. The flow associated with the current is assumed to be self-existent (background current). Moreover, we will restrict ourselves to large-scale currents, where the horizontal and time variations of the background current are small compared to the characteristic wavelength and wave period, respectively. In this framework, a new coupled-mode model is developed for the wave–current–seabed interaction problem, with application to wave scattering by non-homogeneous, steady ambient currents over general bottom topography. Based on previous work by the authors [22–24], the vertical distribution of the scattered wave potential is represented by a series of local vertical modes containing the propagating mode and all evanescent modes, plus additional terms accounting for the satisfaction of the free-surface and bottom boundary conditions. Using the above representation, in conjunction with Luke’s [25] variational principle, the wave–current–seabed interaction problem is reduced to a coupled system of differential equations on the horizontal plane. In the case of small-amplitude waves, a linearised version of the coupled-mode system is also obtained, generalizing previous results by the authors in the case of waves propagating over variable bathymetry regions [22,23]. Keeping only the propagating mode in the vertical expansion of the wave potential, the present system reduces to an one-equation model, that is shown to be compatible with the mild-slope model derived by Kirby [8] with application to the problem of wave–current interaction over slowly varying topography, and exactly reduces to the modified mild-slope equation [26,27] in the case of no current. The present coupled-mode system is discretized on the horizontal plane by using second-order finite differences and numerically solved by iterations. Results are shown for various representative test cases demonstrating the usefulness of the present model, as well as the importance of the first evanescent modes and the additional sloping-bottom mode when the bottom slope is not negligible. The analytical structure of the present model facilitates its extension to fully non-linear waves, and it can be further elaborated to study wave scattering by currents characterised by general 3D structure.

2. Background current flow

We consider wave propagation in the presence of steady, non-homogeneous current, in a variable bathymetry region, Fig. 1. The liquid is assumed inviscid and homogeneous, and the flow associated with the background, nearly horizontal and slowly varying current is assumed to be self-existent, steady and possibly weakly rotational. On the other hand, the wave flow perturbing the background current flow, is generated by an incident wave system coming from the far up-wave region (see Fig. 1), and is assumed to be irrotational. Moreover, the current flow velocity is assumed to be small and, thus, the associated mean free-surface elevation (set-down) is also small. We introduce however, no assumption as regards the mildness of the bottom slope.

A Cartesian coordinate system is used, having its origin at some point on the unperturbed free-surface (z = 0); see Fig. 1. The current field \( \mathbf{q} = (U_1, U_2, W) \) has been assumed to be steady and self-existent, and the kinematics of this flow require that

\[
\nabla \cdot \mathbf{q} = 0, \tag{2.1a}
\]

\[
\mathbf{q} \cdot \mathbf{n} = 0, \quad z = H(x_1, x_2), \tag{2.1b}
\]

\[
\mathbf{q} \cdot \mathbf{n} = 0, \quad z = -h(x_1, x_2), \tag{2.1c}
\]

where \( \nabla = \left[ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right] \) denotes the divergence operator in 3D, \( \mathbf{n} \) is the outward unit normal on the free-surface and bottom boundaries and \( H \) denotes the mean set-down associated with the background current flow. Eqs. (2.1b) and (2.1c) are equivalently written in the form:

\[
W - U_1 \frac{\partial H}{\partial x_1} - U_2 \frac{\partial H}{\partial x_2} = W - \mathbf{U} \cdot \nabla H = 0, \quad z = H(x_1, x_2), \tag{2.2a}
\]
Furthermore, using the fact that the current has been assumed essentially horizontal \( W \ll U_1, U_2 \) and slowly varying \( (|\mathbf{q}|^2 \approx |\mathbf{q}_\infty|^2) \), the above equation can be approximated as follows

\[
P \approx P_0 - \rho g z. \tag{2.6}
\]

The total (current and wave) flow \( \mathbf{v} = \mathbf{q} + \nabla_3 \varphi(x, z; t) \), where \( \varphi(x, z; t) = \psi(x_1, x_2, x_3; t) \) denotes the wave potential, as well as the background current flow \( \mathbf{q} \), both satisfy Euler equations,

\[
\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} \nabla_3 \mathbf{u}^2 - \mathbf{u} \times (\nabla_3 \times \mathbf{u}) = -\frac{\nabla_3 P_0}{\rho},
\]

\[
\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2} \nabla_3 \mathbf{q}^2 - \mathbf{q} \times (\nabla_3 \times \mathbf{q}) = -\frac{\nabla_3 P}{\rho}, \tag{2.7}
\]

where \( P_u \) denotes the total pressure and \( P \) the background (current) pressure. Subtracting these equations by parts and omitting approximately the terms associated with the vorticity of the background flow \( (\nabla_3 \times \mathbf{q}) \) which is assumed to be weak (and noting that the wave disturbance flow is assumed irrotational, i.e., \( \nabla_3 \times \mathbf{u} = \nabla_3 \times \mathbf{q} \)), we obtain

\[
\frac{\nabla_3 P}{\rho} + \frac{\partial \mathbf{q}}{\partial t} + \nabla_3 \mathbf{q}^2 - \frac{1}{2} \nabla_3 \mathbf{q}^2 = 0, \tag{2.8}
\]

where \( p \) denotes the wave-disturbance pressure, defined as the difference between the total \( (P_u) \) and the background \( (P) \) pressure:

\[
p = P_u - P. \tag{2.9}
\]
In addition, using $\nabla \psi = \mathbf{u} - \mathbf{q}$, we obtain $(1/2)\nabla \mathbf{u}^2 - (1/2)\nabla \mathbf{q}^2 = \mathbf{q} \cdot \nabla \psi + (1/2)|\nabla \psi|^2$. Substituting this result in Eq. (2.8) and integrating the following Bernoulli-type equation is obtained

$$\frac{p}{\rho} + \frac{\partial \psi}{\partial t} + \mathbf{q} \cdot \nabla \psi + \frac{1}{2}|\nabla \psi|^2 = 0. \tag{2.10}$$

Combining the above result with Eq. (2.6), we finally obtain the following approximate version of the Bernoulli equation

$$p_u - p_a = -\rho \left[ g z + \frac{\partial \psi}{\partial t} + \mathbf{q} \cdot \nabla \psi + \frac{1}{2}|\nabla \psi|^2 \right], \tag{2.11}$$

expressing the conservation of energy of the studied wave-current problem.

### 3. Variational formulation

Having obtained an expression for the total pressure, we now proceed to the variational formulation and study of the wave-current scattering problem using Luke’s [25] functional, which is based on integration of pressure in the domain (see also [29]),

$$F = \int_0^T \int_{x_1}^{x_2} \int_{z_1}^{z_2} \left( p_u - p_a \right) dz dx dt, \tag{3.1}$$

where $n(x,t) = n(x_1, x_2, t)$ denotes the free-surface elevation associated with the total wave and current flow. Using Eq. (2.11), the above equation is written as follows

$$F = -\rho \int_0^T \int_{x_1}^{x_2} \int_{z_1}^{z_2} \left( g z + \frac{\partial \psi}{\partial t} + \mathbf{q} \cdot \nabla \psi + \frac{1}{2}|\nabla \psi|^2 \right) dz dx dt. \tag{3.2}$$

Requiring the above functional to be stationary, i.e.

$$\delta F = \delta F_\psi + \delta F_\eta = 0, \tag{3.3}$$

where $\delta F_\psi$ and $\delta F_\eta$ denote the first variations of the functional $F$ with respect to the variations of the fields $\psi$ and $\eta$, respectively, and after some algebra we finally obtain the following result:

$$\rho \int_0^T \int_{x_1}^{x_2} \int_{z_1}^{z_2} \left( g v + \left[ \left( \frac{\partial \psi}{\partial t} + U_1 \frac{\partial \psi}{\partial x_1} + U_2 \frac{\partial \psi}{\partial x_2} - \frac{1}{2}\left( \nabla \psi \right)^2 \right) \right] \right) \eta_{x_1} \eta_{x_2} \delta \eta - \left[ \frac{\partial \psi}{\partial x_1} \eta_{x_2} + \frac{\partial \psi}{\partial x_2} \eta_{x_1} \right] \delta \psi + \int_{z_1}^{z_2} \left[ \frac{\partial \psi}{\partial x_1} \eta_{x_2} + \frac{\partial \psi}{\partial x_2} \eta_{x_1} \right] dz \delta \eta = 0 \tag{3.4}$$

Details are provided in Appendix A. From the above result the equations governing the non-linear wave–current scattering problem are derived by exploiting the independence of the variations of the wave potential $\delta \psi$ and the free-surface elevation $\delta \eta$ in the domain and on the various parts of the boundary. More precisely, it is directly seen that the variational equation $\delta_\eta F = 0$ (associated with the $\delta \psi$ – terms of Eq. (3.4)) models the wave kinematics:

$$\nabla^2 \psi + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad (x_1, x_2, z) \in D, \tag{3.5a}$$

$$\frac{\partial \eta}{\partial t} + (\mathbf{U} + \nabla \psi) \cdot \nabla \eta - \left( W + \frac{\partial \psi}{\partial z} \right) = 0, \quad \text{on } z = \eta(x_1, x_2, t), \tag{3.5b}$$

$$\nabla \psi \cdot \nabla h + \frac{\partial \psi}{\partial z} = 0, \quad \text{on } z = -h(x_1, x_2). \tag{3.5c}$$

where $D$ denotes the fluid domain. Furthermore, the equation $\delta_\eta F = 0$ (associated with the $\delta \eta$ – term of Eq. (3.4)) models the free-surface dynamics (Bernoulli’s integral):

$$g \eta + \frac{\partial \psi}{\partial t} + U_1 \frac{\partial \psi}{\partial x_1} + U_2 \frac{\partial \psi}{\partial x_2} + \frac{1}{2}\left( \nabla \psi \right)^2 = 0, \quad \text{on } z = \eta(x_1, x_2, t) \tag{3.6a}$$

It is easily seen that the above equation, using Eq. (2.11) evaluated at $z = \eta$, takes the usual form of the dynamic condition

$$p_e = p + p_a, \quad \text{on } z = \eta(x_1, x_2, t), \tag{3.6b}$$

stating that the total pressure on the free-surface is equal to the atmospheric one.

### 4. Local-mode representation of the wave potential

Following previous works [30–32] we introduce the following enhanced local-mode series expansion of the wave potential

$$\psi(x, z, t) = \psi_{-2}(x, t) \mathcal{Z}_{-2}(z; h, \eta) + \psi_{-1}(x, t) \mathcal{Z}_{-1}(z; h, \eta) + \psi_0(x, t) \mathcal{Z}_0(z; h, \eta), \tag{4.1}$$

in the variable bathymetry region, where $(x = (x_1, x_2))$. In the above expansion, the mode $n = 0$ (corresponding to the term $\psi_0(x) \mathcal{Z}_0(z; \hat{h}, \eta)$) is called the propagating mode and the remaining terms, $n = 1, 2, \ldots$, are the evanescent modes. The term $\psi_{-1}(x, t) \mathcal{Z}_{-1}(z; h, \eta)$ is a correction term, called the sloping-bottom mode, accounting for the satisfaction of the Neumann boundary condition on the non-horizontal parts of the bottom. This term has first been introduced and studied by Athanassoulis and Belibassakis [22] for the linearised water wave problem in variable bathymetry regions. The term $\psi_{-2}(x, t) \mathcal{Z}_{-2}(z; h, \eta)$ is a similar additional term permitting the satisfaction of the non-linear free-surface boundary condition in the case of general wave systems studied in the time-domain; see [32]. A specific convenient form for this term is given by

$$Z_{-2}(z; h, \eta) = \frac{\mu h_0 - 1}{2(\eta + h)h_0}(z + h)^2 - \frac{\mu h_0 + 1}{2h_0}(\eta + h) + 1. \tag{4.2}$$

satisfying $Z_{-2}(z = -\eta) = \mu_0 Z_{-2}(z = \eta) = 1/\mu_0, \quad Z_{-2}(z = -h) = 0$, where the prime denotes differentiation with respect to $z$. As

$$Z_{-1}(z, h, \eta) = \frac{\mu_0 h_0 - 1}{2(\eta + h)h_0}(z + h)^2 + \frac{1}{h_0}(z + h) + \frac{2h_0 - (\eta + h)(\mu_0 h_0 + 1)}{2h_0}, \tag{4.3a}$$

satisfying $Z_{-1}(z = -\eta) = \mu_0 Z_{-1}(z = \eta) = Z_{-1}(z = -h) = 1/\mu_0$ in Eqs. (4.2) and (4.3a) the parameters $\mu_0, h_0 > 0$ are positive constants treated as numerical parameters. However, other possible forms concerning the above additional modes are possible. For example, as concerns the sloping-bottom mode, the function defined by

$$Z_{-1} = (h + \eta) \left( \frac{z - \eta}{h + \eta} \right)^3 + \left( \frac{z - \eta}{h + \eta} \right)^2, \tag{4.3b}$$

satisfying similar as before conditions and $Z_{-1}(z = 0) = Z_{-1}(z = 0) = 0$ can serve for the same purpose (see also [22]).
The rest (infinite set) of the vertical modes $Z_n(z)$, $n = 0, 1, 2 \ldots$, involved in the expansion (4.1), are obtained as the eigenfunctions of local, vertical Sturm–Liouville problems formulated with respect to the local depth and an intrinsic-frequency type parameter

$$\sigma = \omega - U \cdot \nabla \phi$$  \hspace{1cm} (4.4)

where $\omega$ denotes a reference (constant) value of the absolute wave frequency (e.g., taken as the central frequency of the wave) in the domain, and $\nabla \phi(x, t)$ contains information concerning the phase of wave motion which is assumed to take place essentially on the horizontal plane. The functions $Z_n(z)$ are generated as the solution of the following vertical problem formulated at each horizontal position $x$,

$$Z''_n + \kappa^2 Z_n = 0, \quad -h(x) < z < \eta(x, t),$$  \hspace{1cm} (4.5a)

$$Z'_n(z = \eta) - \mu Z_n(z = \eta) = 0, \quad z = \eta(x, t),$$  \hspace{1cm} (4.5b)

$$Z'_n(-h(x)) = 0, \quad z = -h(x),$$  \hspace{1cm} (4.5c)

where $\mu = \sigma^2 |g|$ is the corresponding frequency parameter. The parameter $\sigma$ depends on the horizontal field $\nabla \phi$, a quantity that is associated with the wave kinematics (see, e.g., [4]). Thus, both $\sigma$ and $\mu$ are dependent on the solution $\phi(x, z, t)$ of the problem, which introduces additional intrinsic nonlinearity to the wave–current scattering problem, and iterations are necessary for their determination (even in the case of the linearised version examined in more detail below in Section 6). However, at each time instant, both $\nabla \phi$ and $\mu = \sigma^2 |g|$ are considered to be known, horizontally variable real functions, which enables us to construct, through the solution of Eq. (4.4), local (at each horizontal position $x, y$) real vertical bases $Z_n(z)$, that are parametrically dependent on time:

$$Z_0 = \frac{\cos[k_0(z + h)] - \cos[k_0(\eta + h)]}{\cos[k_0(\eta + h)]}, \quad Z_n = \frac{\cos[k_n(z + h)] - \cos[k_n(\eta + h)]}{\cos[k_n(\eta + h)]}, \quad n = 1, 2, \ldots, n \geq 1,$$  \hspace{1cm} (4.6a)

where the wavenumbers $k_0 = ik_0(h, \eta), k_n = k_n(h, \eta), n \geq 1$, are obtained as roots of the local ‘dispersion-type’ relation (s):

$$\sigma^2 = k_0 g \tanh[k_0(h + \eta)], \quad \sigma^2 = k_n g \tan[k_n(h + \eta)], \quad n \geq 1,$$  \hspace{1cm} (4.6b)

and $i = \sqrt{-1}$. Note that at each iteration (or time step) the parameter $\sigma$ defined by Eq. (4.4) is treated as a known function of the two horizontal variables. As it will be discussed in more detail in the sequel, the validity of the expansion (4.1) is based on the completeness properties of the regular Sturm–Liouville systems, as (4.5) in the vertical interval $-h(x) < z < \eta(x, t)$. We wish to stress here the fact that, in the general case of waves scattered by inhomogeneous current in variable bathymetry, which is the subject of the present work, the phase gradient $\nabla \phi$ is not equal with any of the eigenvalues $k_n, n > 0, 1, 2 \ldots$ and thus, the system (4.6a) maintains at all times and all horizontal positions the basis properties of regular eigenfunctions [33].

However, in the case of linearised monochromatic wave motion, that will be examined in more detail in Section 6, under the additional assumption of very slowly varying depth and current, we obtain from the solution of the problem that the first eigenvalue $k_0$ comes close, but never becomes equal, to the gradient of the wave phase ($k_0 \approx |\nabla \phi|$). In this case, the asymptotic treatment of Eq. (4.6b) for $k_0$ indicates that this equation may have none or one or two roots, depending on the sign and the magnitude of $U \cdot \nabla \phi$ (as, e.g., discussed by Mei [1], Section 3.7, in the analysis of waves scattered by uniform, following or opposing, current in horizontal strip). Interested in the case of waves that maintain their physical essence as they propagate in the presence of currents in general bathymetry, lengthened or shortened by the following or opposing current (as depicted by points A and C, in Fig 7.1 of Ref. [1], respectively), we proceed by considering $U \cdot \nabla \phi / \omega$ to be a relatively small quantity with respect unity, and thus, $\sigma$ to be positive and close to $\omega$.

The validity of the expansion (4.1) concerning the representation of the wave potential is discussed in Refs. [31,32]. We remark here that the sloping-bottom mode ($\phi \propto Z_{-1}$) is zero

$$\phi_{-1}(x, t) = 0,$$  \hspace{1cm} (4.7)

and thus, it is not needed, in subareas where the bottom is flat ($\nabla h = 0$). Moreover, the upper-surface mode ($\phi \propto Z_{+1}$) becomes zero only in the very special case of linearised (small-amplitude) monochromatic waves scattered by uniform (follow or opposing) current characterised by frequency parameter $\mu$ that coincides with the choice of numerical parameter $\mu_0, (\mu = \mu_0)$.

The series expansion (4.1) permits us to obtain corresponding series representations for the wave potential and its derivatives, as well as for the variation $\delta \phi$ of the wave potential in terms of the variations of the modal amplitudes $\delta \eta_n$ and the upper surface elevation $\delta \eta$. The latter substituted in the variational principle (3.4) permits us to obtain a reformulation of the problem in the form of a non-linear coupled-mode system (CMS) with respect to the modal amplitudes $\delta \eta_n$ and the upper surface elevation $\delta \eta$.

In the case of no-current ($U = \nabla \phi = 0$) the study and numerical solution of the above coupled-mode system associated with the propagation and scattering of nonlinear water waves in general bathymetry regions has been presented in [30,32,34]. The derivation of steady travelling non-linear wave solutions in constant depth strips has been presented in detail in [32], where also comparisons are presented with high-order Stokes and cnoidal wave theories, respectively, in intermediate and shallow water depth. Finally, linearised and second-order versions of the above CMS (without current) have been studied in [23,31]. Illustrating that the dispersion characteristics of the present model using 5–6 terms in the expansion (4.1) agree quite well with linearised and second-order wave theory, for non-dimensional wavenumbers in the interval $0 < kh < 24$.

In the next section we will present and discuss the application of the present CMS in the case of linear harmonic waves scattered by non-uniform horizontal currents, in general bottom topography. The detailed analysis of the correspondence of non-linear coupled-mode system is left to be subject of a future work.

5. The linearised CMS for harmonic wave motion

By assuming small wave amplitudes and considering the wave velocity field to be time harmonic with angular frequency $\omega$, the latter is described by means of a complex velocity potential in the form:

$$\phi(x_1, x_2, z, t) = \text{Re}(\psi(x_1, x_2, z) \exp(-i\omega t)).$$  \hspace{1cm} (5.1a)

In this case, by omitting the explicit quadratic and high-order terms and linearising, the complex amplitude of the free-surface elevation is obtained from Eq. (3.6a) in terms of the corresponding values of the complex wave potential as follows

$$\eta = -\frac{1}{g} \left(-k \omega - U_t \frac{\partial}{\partial x_1} + U_2 \frac{\partial}{\partial x_2}\right) \psi(x_1, x_2, z = 0).$$  \hspace{1cm} (5.1b)

Also, truncating the upper end of vertical integration interval $-h(x) < z < \eta(x, t)$ from $z = \eta(x, t)$ to $z = 0$, i.e., up to the point corre-
sponding to the unperturbed free surface, the variational principle (3.4) simplifies considerably, and becomes

\[
\rho \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left\{ \left( -\frac{i\omega}{g} \mathbf{V} \cdot \mathbf{U} \right)^2 + \frac{1}{g} (\mathbf{V} \cdot \mathbf{U}) \right\} dz \delta \varphi - \frac{\partial \varphi}{\partial z} \bigg|_{z=0} + \frac{\delta \varphi}{\delta \varphi} \bigg|_{z=h} \delta \varphi + \int_{z=h}^{z=0} \left( \frac{\partial^2 \varphi}{\partial z^2} \right) dx \delta \varphi = 0,
\]

(5.2)

where from now on, \( \varphi = \varphi(x, z) \) denotes the complex wave potential and \( \mathbf{U}(x) \) the horizontal current on \( z = 0 \). Noticing that the term

\[
A = \left( -\frac{i\omega}{g} \mathbf{V} \cdot \mathbf{U} \right)^2 + \frac{1}{g} (\mathbf{V} \cdot \mathbf{U}) \varphi,
\]

(5.3a)
on the mean free-surface (\( z = 0 \)) appearing in the above integral can also be written as follows:

\[
A = -\partial^2 \varphi - \frac{2i\omega}{g} (\mathbf{V} \cdot \mathbf{U}) \varphi - i\omega (\mathbf{V} \cdot \mathbf{U}) \varphi + \mathbf{U} \cdot (\mathbf{U \cdot V}) \varphi, \tag{5.3b}
\]

the variational principle (5.2) describing small-amplitude, time–harmonic wave motion is finally put in the form:

\[
\int_{x_1}^{x_2} \int_{y_1}^{y_2} \left\{ \int_{z=h}^{z=0} \left( \frac{\partial^2 \varphi}{\partial z^2} \right) dx \delta \varphi \right\} dx dy + \frac{\delta \varphi}{\delta \varphi} \bigg|_{z=h} \delta \varphi - \frac{\partial \varphi}{\partial z} \bigg|_{z=0} + \frac{\partial \varphi}{\partial z} \bigg|_{z=0} = 0.
\]

(5.4)

In the present case, the mode \( \varphi_{-2}(x, t) \) associated with the satisfaction of the free-surface boundary condition is not needed, since the parameter \( \rho \) can be suitably chosen to coincide with the frequency parameter \( \mu \). Thus, the local-mode series expansion of the complex wave potential attains the following simplified form

\[
\varphi(x, z) = \sum_{n=1}^{\infty} \varphi_n(x) \cdot Z_n(z),
\]

(5.5)

where the functions \( Z_n(z) = Z_n(z; h(x), \eta = 0) \), \( n = 0, 1, 2, \ldots \), are now obtained as the eigenfunctions of local vertical Sturm–Liouville problems (4.5), formulated with respect to the local depth and the local intrinsic frequency \( \sigma = \omega - \mathbf{U} \cdot \mathbf{V} \) in the vertical interval \( h(x) < z \) \( < 0 \). These local vertical eigenfunctions are given by

\[
Z_0 = \frac{\cosh[k_0(z + h)]}{\cosh(k_0 h)}, \tag{5.6a}
\]

\[
Z_n = \frac{\cosh[k_n(z + h)]}{\cosh(k_n h)}, \quad n = 1, 2, \ldots , \tag{5.6b}
\]

and have the property to be scaled, i.e., \( Z_0(z = 0) = 1, n = 0, 1, 2, \ldots \), and as the sloping-bottom modes \( \varphi_{-1}, \varphi_{-2}, \ldots \), a specific alternative form of the function \( Z_{-1}(z) \) used in this case is defined by

\[
Z_{-1}(z) = h \left[ \left( z \over h \right)^3 + \left( z \over h \right)^2 \right], \tag{5.8}
\]

h
having the following properties: \( Z_{-1}(z = -h) = 1, Z_{-1}(z = -h) = 0 \), and \( Z_{-1}(z = 0) = Z_{-1}(z = 0) = 0 \) (see also the discussion in Ref. [22]). Using the representation (5.5), in conjunction with the properties of \( Z_n \), Eqs. (5.6), we obtain

\[
\frac{\partial \psi}{\partial z} \bigg|_{z=0} = \mu \psi = \frac{\sigma^2}{g} \psi, \quad \text{on } z = 0.
\]

(5.9)

Introducing the above result in the last form of the variational principle, Eq. (5.4), the latter reduces to

\[
\int_{x_1}^{x_2} \int_{y_1}^{y_2} \delta \varphi + \int_{z=h}^{z=0} \left( \frac{\partial^2 \varphi}{\partial z^2} \right) dx \delta \varphi = \int_{z=h}^{z=0} \left( \frac{\partial^2 \psi}{\partial z^2} \right) dx \delta \psi
\]

\[
+ \left[ \frac{\partial \varphi}{\partial z} + \mathbf{V} \cdot \mathbf{U} \right] \delta \varphi - \frac{\partial \varphi}{\partial z} \bigg|_{z=0} = \frac{\partial \varphi}{\partial z} \bigg|_{z=0} = 0. \tag{5.10}
\]

5.1. The coupled-mode system for linearised waves

Introducing the local-mode representation (5.5) in the variational equation (5.10), and expressing the variation \( \delta \varphi \) of the wave potential in terms of the variations of the modal amplitudes \( \delta \varphi_n \), we finally obtain the following coupled-mode system (CMS) of equations, associated with the linearised wave–current problem:

\[
\sum_{n} \left\{ a_{mn} \nabla^2 \varphi_n + \left[ b_{mn} + \frac{2i\omega}{g} \mathbf{U} \right] \nabla \varphi_n + \left[ c_{mn} + \frac{\omega^2 - \sigma^2}{g} \right] \varphi_n \right\} = 0, \quad m = -1, 0, 1, \ldots , \tag{5.11}
\]

In the above system, the coefficients \( a_{mn}, b_{mn}, c_{mn} \) are given by

\[
a_{mn} = \left( Z_n, Z_m \right), \tag{5.12a}
\]

\[
b_{mn} = 2 \left( \nabla Z_n, Z_m \right) + Z_n(-h)Z_m(-h)\nabla h, \tag{5.12b}
\]

\[
c_{mn} = \left( \nabla^2 Z_n + \nabla \cdot \left( \frac{\partial^2 Z_n}{\partial z^2} \right) Z_m(-h) + \left( \frac{\partial Z_n}{\partial z} \right) Z_m(-h) \right), \tag{5.12c}
\]

where (from now on) the brackets denote the inner product \( \left( f_1, f_2 \right) = \int_{z=-h}^{z=0} f_1(z)f_2(z)dz \) in the vertical interval \( -h < z < 0 \). The above system of equations is essentially linear with respect to the unknown mode amplitudes \( \varphi_n(x, t) \), \( n = -1, 0, 1, 2, \ldots \). Still, however, the intrinsic frequency depends on the phase gradient \( \nabla \Psi \) which is defined in terms of the solution \( \varphi(x, z) \) as follows,

\[
\Psi(x) = \arg(\varphi(x, z = 0)) = \arg \left( \sum_{n=0}^{\infty} \psi_n(x) \right), \tag{5.13}
\]

where \( \arg \) denotes the argument of complex number. Thus, iterations are necessary for the determination of \( \sigma \) (and the corresponding intrinsic frequency parameter \( \mu \)) which are involved in the calculation of the vertical eigenfunctions \( Z_n(z), n = 0, 1, 2, \ldots \), and the coefficients of the CMS (5.11) and (5.12)). It is clear from Eq. (5.13) that all modes \( \varphi_n(x) \) contribute to the phase \( \Psi(x) \) and thus, in our case (general bathymetry and inhomogeneous current) both \( \nabla \Psi \) and the intrinsic frequency \( \sigma \) do not explicitly involve anyone of the vertical eigenvalues \( k_n, n = 0, 1, 2, \ldots \), which do not enter in the boundary condition of the vertical problem (4.5b) on the free surface. Thus, at any horizontal point in the domain, the set of functions (5.6) constitutes a local vertical basis exploited for the expansion of the wave potential through Eq. (5.5).

Finally, we remark here that in the case of no-current (\( \mathbf{U} = W = 0 \)) the above CMS reduces exactly to the corresponding system presented by the authors [22] for studying linearised wave propagation over variable bathymetry regions.

5.2. One-equation model for wave scattering by current

A significant simplification of the CMS (5.11) is obtained by keeping only the propagating mode \( n = 0 \) in the local-mode series
The expansion of the wave potential (5.5), which essentially describes the propagation features. In this case, the above coupled-mode system (5.11), (5.12) is reduced to the following one-equation model on the horizontal plane,

$$a_{00} \nabla^2 \psi_0 + \left( b_{00} + \frac{2i\omega}{k} \right) \cdot \nabla \psi_0 - \frac{1}{g} \nabla \cdot (U[U \cdot \nabla] \psi_0) + \left[ c_{00} + \frac{\omega^2 - \sigma^2}{g} + \frac{i\omega}{g} (\nabla \cdot \mathbf{U}) \right] \psi_0 = 0$$  \hspace{1cm} (5.14)

The coefficients $a_{00}, b_{00}, c_{00}$ in the above equation are provided by Eqs. (5.12) for $m = n = 0$, and after some processing they become as follows:

$$a_{00} = \int_{z=-h}^{z=0} Z_0^2 dz = \frac{1}{2k_0} \tanh(k_0 h) \left( 1 + \frac{2k_0 h}{\sinh(2k_0 h)} \right) = \frac{1}{g} CC_g,$$

$$b_{00} = \nabla a_{00} = \frac{1}{g} \nabla CC_g = 2 \left( \nabla Z_0, Z_0 \right) + \left( \nabla Z_0 (z = -h) \right) h, \quad (5.15b)$$

$$c_{00} = k_0^2 a_{00} + \left( \nabla^2 Z_0, Z_0 \right) + \left( \nabla Z_0 (z = -h) \right) Z_0 (z = -h) - \left( 2\omega C_0 + \gamma \right), \quad (5.15c)$$

where $C = \sigma/k_0(\sigma)$ and $C_0 = d\sigma/dk_0$ denote the local wave phase and group velocities relative to the current, respectively, calculated at the intrinsic frequency $\sigma$.

The coefficient $c_{00}^{(2)}$ contains terms proportional to first and second horizontal derivatives of the depth function (proportional to bottom slope and curvature), as well as first and second horizontal derivatives of the horizontal current velocity components $U_1$ and $U_2$. Using the above expressions (5.15) of the coefficients in the one-equation model (5.14) and multiplying it by $g$, it is easily seen that the latter takes the form

$$\nabla (CC_g \nabla \psi_0) - \nabla \cdot (U[U \cdot \nabla] \psi_0) + 2i\omega U \cdot \nabla \psi_0 + \left( k_0^2 CC_g + g \sigma \right) + \omega^2 - \sigma^2 + i\omega (\nabla \cdot \mathbf{U})] \psi_0 = 0,$$

which will be called here the modified mild-slope equation for wave scattering by ambient horizontal current in general bottom topography. A similar result has been presented by Hsu et al. [35] based on the analysis by Dingemans [36]. We note here that the above model (5.16) in the case of no current ($U = 0$), reduces exactly to the modified mild slope equation derived by Massel [26] and Chamberlain and Porter [27]. In this case, Eq. (5.14) takes the form

$$\nabla (a_{00} \nabla \psi_0) + (k_0^2 a_{00} - K) \psi_0(x) = 0$$  \hspace{1cm} (5.17a)

where

$$K = -c_{00}^{(2)} - \int_{z=-h}^{z=0} (\nabla^2 Z_0(z; x)) Z_0(z; x) dz - \left( \nabla Z_0 (z = -h) \right) h \nabla Z_0 (z = -h) h: x = K_1 \nabla^2 h + k_0 K_2 (\nabla h)^2,$$

and $K_1, K_2$ are functions of $k_0 h$, as given by Miles and Chamberlain (37), Eqs. (1.14b, c). Furthermore, we remark here that if the term $c_{00}^{(2)}$ is omitted, then Eq. (5.16) becomes

$$\nabla (CC_g \nabla \psi_0) - \nabla \cdot (U[U \cdot \nabla] \psi_0) + 2i\omega U \cdot \nabla \psi_0 + \left( k_0^2 CC_g + \omega^2 - \sigma^2 + i\omega (\nabla \cdot \mathbf{U}) \right) \psi_0 = 0,$$

which is exactly the mild slope equation derived by Kirby [8], in its time-harmonic form; see also Chen et al. [11]. We note here that the derivation of (5.18) is also presented in Dingemans [36] using the Hamiltonian formulation.

We conclude this section by noticing that in the simple 1D-horizontal case, concerning scattering of waves by current of magnitude $U(x)$, and intrinsic frequency $\sigma = \omega - qkU$, where $q = 1$ corresponds to flow and $q = -1$ to opposing current, respectively, flowing in the same direction (along the $x$-axis) as the wave propagation, our model (5.16) takes the form

$$\frac{d}{dx} \left[ \frac{CC_g - U^2}{2} \frac{d\psi_0}{dx} + 2i\omega U \frac{d\psi_0}{dx} + [k_0^2 CC_g + g \sigma \psi_0 + \omega^2 - \sigma^2] \psi_0 = 0. \right.$$

The above 1D equation could be found useful in studying waves scattered by unidirectional current, as e.g., in the case of open channel flow. In this case, the coefficient $c_{00}^{(2)}$ can be analytically calculated

$$c_{00}^{(2)} = \int_{z=-h}^{z=0} \frac{\partial^2 Z_0(z; x)}{\partial x^2} \left( 24k_0^4 \cos^2 k_0 h \right)^{-\frac{1}{2}} \left( \frac{dZ_0}{dx} \right)^2 \frac{dZ_0(z; x)}{dx} dz + h' \left( \frac{dZ_0}{dx} \right)^2 \left( z = h \right), \quad (5.20)$$

where

$$\beta = \int_{z=-h}^{z=0} \left( \frac{dZ_0(z; x)}{dx} \right)^2 \frac{dZ_0(z; x)}{dx} dz + h' \left( \frac{dZ_0}{dx} \right)^2 \left( z = h \right),$$

and the terms $v_1 = 6k_0^4 h (2k_0 h + \sinh(2k_0 h)), v_2 = 6k_0^4 h (-3 + \cosh(2k_0 h) \sinh(2k_0 h)), v_3 = 6k_0^4 h (2k_0 h + \sinh(2k_0 h) + 2\cosh(2k_0 h) \sinh(2k_0 h))$. Moreover, $\gamma = (dx/dx - \sec^2 h (2k_0 h) dh/dx)$ and $\sigma = \omega_0$. In the above equations $Z_0(z; x)$ is given by Eq. (5.6a) and $k_0$ is the positive root of the dispersion relation $\mu h = k_0 \tan(h(k_0 h), Eq. (5.7), for $n = 0$. From the latter expression, the derivative $dk_0/dx$ can be also calculated, and is given by:

$$\frac{dk_0}{dx} = -k_0 2g \sigma (U/dx) + k_0 g \sec h^2(k_0 h) (dh/dx) + \frac{g k_0 h \sec h^2(k_0 h) + \tan(h(k_0 h) + 2g \sigma U}{g k_0 h \sec h^2(k_0 h) + \tan(h(k_0 h) + 2g \sigma U} \right).$$

6. Numerical results and discussion

The present coupled-mode system (5.11) is discretized by using second-order, central finite differences and numerically solved by means of a parallel implementation of the model; see, e.g., [38]. Thus, direct numerical solution of the present system is made feasible for realistic domains corresponding to areas with size of the order of many characteristic wavelengths. For the numerical solution of the problem the system of Eq. (5.11) is supplemented by appropriate boundary conditions expressing the incoming wave, at the specific direction for given frequency, along the offshore boundary (region of incidence). Furthermore, all along the lateral boundaries and the shoreward boundary (region of transmission) absorbing boundary conditions are imposed, using the perfectly matched layer (PML). More details concerning the implementation of the PML are given in [23].

The present numerical solution is obtained by iterations that are necessary to determine the distribution of the intrinsic frequency $\sigma$ all over the domain, which depends on the local phase gradient $\nabla \psi$ of the wave, as calculated through Eq. (5.13) and involving the wave potential on the free surface. The latter quantity provides us information concerning the local wave direction with respect to the background current flow. The iterative procedure to treat the above
implicit non-linearity starts by assuming an initial information concerning the wave field and the wave direction, which is calculated by means of the present CMS (5.11) by totally neglecting the current. Then, a first estimation of $\sigma$ is obtained by means of Eq. (4.4), through which the local eigenvalues and vertical eigenfunctions and the coefficients of the CMS are calculated. Next, the above system is again solved taking into account the current and a new wave field is computed, and iterations are repeated until convergence. In this way, the intrinsic frequency and all dependent quantities, i.e., the local vertical eigenvalue problem, the coefficients and finally the coupled-mode system, essentially carry out Doppler effects in all cases of following, opposing and more general current conditions, as it will be illustrated in the following examples.

To demonstrate the applicability and usefulness of the present method, in this section, numerical results are presented and discussed in the case of 3D and 2D wave propagation over variable bottom topography regions. Four representative test cases have been considered, corresponding to a rip current over a steep sloping bed in shallow water, scattering by a ring current in constant depth, study of wave reflection by sinusoidal bathymetry in the presence of current and scattering by current generated due to breaking of waves above an elliptic shoal.

6.1. Waves scattered by rip current over a steep sloping bed

Wave-induced rip currents, created by longshore flow fields converging into periodic rips and forming independent circulation cells in the inshore region, play an important role in coastal morphodynamics. Also, rip currents, in conjunction with local amplification of wave energy, are responsible for many accidents in beaches. In order to illustrate the effects of a rip current, in conjunction with abrupt changes of the bathymetry, on the wave scattering, we examine a steep shoaling area shown in Fig. 2. Using similar expressions as the ones discussed in Chen et al. [11], the rip–current field $\mathbf{U} = (U_1, U_2)$ considered in this example is modelled as follows:

$$U_1 = -0.0681 x_1 F \left( \frac{x_1}{76.2} \right) F \left( \frac{x_2}{76.2} \right),$$

$$U_2 = -0.256 \left[ 2 - \left( \frac{x_1}{76.2} \right)^2 \right] F \left( \frac{x_1}{76.2} \right) \int_{\tau=0}^{x=x_0/7.62} F(\tau) d\tau,$$

where $F(\tau) = (2\pi)^{-1/2} \exp(-\tau^2/2)$. The structure of the current is also shown in Fig. 2 and its maximum value is selected to be 0.5 m/s. In this example, in order to demonstrate the effects of the present multi-modal solution we consider a smooth but steep bottom profile connecting two constant depth regions with depths $h_1 = 4$ m and $h_2 = 0.5$ m, respectively. The variable bathymetry containing the inhomogeneous current is characterized by the following depth function

$$h(x_1) = \frac{h_1 + h_2}{2} - \frac{h_1 - h_2}{2} \tanh(0.2(x_1 - 200)),$$

and essentially extends from $x = 150$ m to $x = 250$ m. The mean bottom slope is 12% and the maximum bottom slope (at $x = 200$ m) is 35%. We consider incident harmonic waves of period $T = 14$ s (corresponding to swell) propagating along the $x_1$-axis (normally to the bottom contours); see Fig. 2. In this case, the phase velocity of the incident wave is $C = 5.5$ m/s. The scattered wave field is calculated by means of the solution of the present coupled-mode system using 5 modes (the propagating mode $n = 0$, the first evanescent modes $n = 1, 2, 3$ and the sloping-bottom mode $n = -1$), which were found to be enough for convergence (error less than $10^{-3}$). In the examined case, a 350 m by 200 m horizontal domain is considered, discretized by using a uniform gridsize of 1 m along each horizontal direction. The number of gridpoints per wavelength ranges from about 86 (in the deeper region) to 31 (in the shallower region) which is considered enough for an accurate numerical solution.

The real part of the calculated wave field on the horizontal and vertical planes, normalized with respect to the amplitude of the incoming wave potential, is plotted in Fig. 2. Also, in the last subplot the bottom profile is shown, as well the calculated wave field (using equipotential lines) and a scaled (exaggerated) distribution of the wave potential on the free-surface, which is connected to free-surface elevation; cf. Eq. (5.11b). In this example, significant focusing of wave energy is observed as the shallow-end of the variable bathymetry region is approached, taking place immediately.
of wave refraction and diffraction are carried out by the propagating mode. In the examined case, we observe that the first evanescent mode and the sloping-bottom mode are of the order of 2–3% of the propagating mode. Also, these modes are clearly confined within the subdomain containing the inhomogeneities (variable bathymetry and current). Outside the limits of the variable bathymetry region \((x_1 < 100 \text{ m}, x_1 > 250 \text{ m})\) the first evanescent mode presents an exponential decay behaviour. On the other hand, the sloping-bottom mode becomes identically zero in these subregions; cf. Eq. (4.7).

The above facts, in the case of environments characterised by mild inhomogeneities, fully justify the truncation of the local-mode series (5.5) keeping only its first term \((n = 0)\) corresponding to the propagating mode, reducing thus, the coupled-mode system (5.11) to the modified mild-slope equation (5.14) or (5.16). However, in the case of steeper bottom topographies and/or more rapidly varying currents, the rest modes become significant and should be retained. As demonstrated in [22,23,39] in the case of variable bathymetry regions, the rest of the modes, and especially, the sloping bottom mode play an important role in satisfying the bottom boundary condition, requiring the normal derivative of the wave potential to vanish there. Also, the inclusion of this term \(\psi_{-1}(x_1, x_2)\) substantially accelerates the rate of convergence of the present local-mode series, permitting us to obtain accurate results for bottom slopes up to and exceeding 100% by keeping only the few first \((3–5)\) terms in the expansion.

### 6.2. Waves scattered by a vortex-ring current

In this example we consider wave scattering by a vortex ring current in constant depth \(h = 10 \text{ m}\), examined by Chen et al. [11]. Such current structures are observed in open sea and coastal areas, and could have important impact on physical and biological processes in the region; see [40]. Following [41], the background current flow associated with the vortex ring is defined by,

\[
U_r = 0, \quad U_\theta = \begin{cases} 
C_S (r/R_1)^2, & r \leq R_1 \\
C_S \exp[-(R_2 - r)^2/R_2^2], & r > R_1
\end{cases}
\]

where \(U_r\) and \(U_\theta\) denote the radial and tangential components of the horizontal flow \(\mathbf{U}\), in a cylindrical-polar coordinate system \((R = |\mathbf{x}|, \quad \theta = \tan^{-1}(x_2/x_1))\) with origin at the center of the vortex ring; see Fig. 4.

The following values of the parameters (see [40]) have been selected to describe the present vortex-ring example: \(C_S = 0.9 \text{ m/s}, \ C_\theta = 1.0 \text{ m/s} \) and \(R_1 = 343.706 \text{ m}, R_2 = 384.881 \text{ m}, R_3 = 126.830 \text{ m}\). The above rotational flow creates a horizontal current with maximum tangential velocity 1 m/s. We consider incident harmonic waves of period \(T = 20 \text{ s}\), propagating along the \(x_1\)-axis, with phase velocity \(C = 9.74 \text{ m/s}\), which are scattered by the above vortex-ring current in constant depth.

In the examined case, an almost 4 km by 4 km horizontal domain is considered, which is discretized by using 301 equidistant points along each horizontal direction, corresponding to about 15 gridpoints per wavelength. Numerical results obtained by the present model are shown in Fig. 4 as concerns the real part of the wave field on the free surface (normalised with respect to the amplitude of incoming wave potential). In this case the bottom is flat and the sloping bottom mode is zero. We clearly observe in this figure the spatial focusing and defocusing of wave energy in the area downwave the vortex ring features that are very well reproduced. Moreover, the propagating mode \(\psi_0(x_1, x_2)\) carries almost all information concerning the spatial wave evolution, and this term can be also obtained by numerically solving the modified mild-slope equation (5.14) or (5.16).

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**Fig. 3.** Modal amplitudes (moduli) of the scattered wave field by a rip current in a sloping beach region. (From top to bottom): the propagating mode \((n = 0)\), the first evanescent mode \((n = 1)\) and the sloping bottom mode \((n = -1)\).
A comparison concerning the calculated distribution of free surface elevation, along various transects downwave the vortex ring, as obtained by the present method against numerical results presented by Chen et al. [11] which are based on Kirby’s model, Eq. (5.18), is shown in Fig. 5. It can be seen that in the examined case the present CMS solution is in reasonable agreement with predictions by the latter model. Small differences are observed in the first transect, near the center of the vortex ring, where the present model predicts higher wave amplitudes, that is considered as an indication of increased diffraction prediction. The corresponding differences appearing in the last transect, in the far-downwave region of the vortex core, are attributed to possible contamination of the present numerical solution there by the absorbing layer which surrounds the computational domain.

6.3. Wave scattering by sinusoidal bathymetry in the presence of current

A detailed experimental and theoretical investigation of wave scattering over a sinusoidal bottom in the presence of an ambient current is presented by Magne et al. [42]. Waves were generated in a flume with and without current, and propagated over a patch of four sinusoidal bars before dissipating on a beach. The mean depth over the sinusoidal patch is $h = 1.5$ m. The amplitude of the bars is 0.4 m with a horizontal bottom wavelength $\lambda_0 = 2.5$ m. The sinusoidal patch is 10 m long including four bottom periodic cells, and is extended by 3 m-long adaptor ramps at the beginning and at the end, to allow for a smooth transition. The particular bottom profile is illustrated in Fig. 6, along with the calculated free-surface elevation corresponding to a harmonic incident wave, as calculated by the present coupled-mode method using a total number of 5 modes, including the propagating mode ($n = 0$), the sloping-bottom mode ($n = -1$) and the first few evanescent modes ($n = 1, 2, 3$).

A comparison of the present method results vs. experimental measurements for the reflection coefficient (from Ref. [42]) is presented in Fig. 7, without and with the effects of a homogeneous horizontal current flowing in the positive x-direction with velocity $U = 0.4$ m/s. Results are presented as a function of the normalized resonant wave number $2k_{nc}/h_0$ defined in accordance with the well known Bragg resonance condition for triad resonant interaction between two waves of wave number $k$ propagating in opposite directions in a homogeneous medium of spatial periodicity $\lambda_0 = 2\pi/h_0$; see Refs. [42,43]. Here, $k_{nc}$ is the wave number corresponding to the mean water depth $h = 1.5$ m for a gravity wave of angular frequency $\omega$ without current, so that $\omega = (gk_{nc} \tan(h_{nc}h))^{1/2}$. In this particular case, the wave period corresponding to Bragg resonance is $T_p = 1.83$ s.
Fig. 6. Normalized free-surface elevation above the sinusoidal bottom profile as calculated by the present CMS, for an incident harmonic wave of period \( T = 1.8 \) s, without (top subplot) and with (lower subplot) current effects. The real part of the free-surface elevation is indicated by using a solid line and the imaginary part by a dashed line, respectively.

Results obtained by the present method which are shown in Fig. 6 correspond to incident wave conditions \( (T = 1.8 \) s) quite close to resonance \( (T_p = 1.83 \) s). Comparing the calculated free-surface elevation above the sinusoidal bottom without current (top subplot of Fig. 6) with the corresponding results with uniform current of velocity \( U = 0.4 \) m/s (lower subplot of Fig. 6), we can clearly observe the strong modification of the wave field all over the domain due to the presence of the current, leading also to significant increase of the reflected energy. We conclude from this example that the present method is able to treat wave scattering in both cases of following, and opposing current conditions.

The reflection coefficient as predicted by the present CMS, for values of the normalized resonant wave number \( 2k_{nc}/\eta_0 \) ranging from 0.6 (corresponding to intermediate water-depth conditions) up to 1.5 (corresponding to almost deep water conditions), is plotted in Fig. 7 concerning the sinusoidal bottom profile. Both cases with and without the effects by the current are considered. Present method results have been obtained keeping 5 modes \( (n = -1, 0, 1, 2, 3) \) in the local-mode series expansion, as well as by keeping only the propagating mode \( (n = 0) \), and are plotted in Fig. 7 by using thick solid lines and dashed lines, respectively. Experimental data from Magne et al. [42] are also shown in the same figure, using crosses in the case of no current and bullets in the case of current \( (U = 0.4 \) m/s), respectively. We remark in the case of no current that the predictions by the present CMS are in excellent agreement with corresponding ones obtained by the multi-mode model by Rey [44] which is based on stepwise approximation of the bathymetry (and compared also in Ref. [42, Fig. 4] against the above experimental data). We observe in Fig. 7 that the present model provides quite reasonable predictions with respect to the experimental measurements in the whole range of frequencies. We also notice in the same figure the effect of evanescent and the sloping bottom modes on the reflection coefficient which becomes more pronounced for lower frequency (i.e., intermediate wave conditions), as expected.

6.4. Waves over an elliptical shoal in the presence of breaking-induced current

In this example we consider one of the test cases presented by Vincent and Briggs [45] concerning refraction/diffraction of irregular waves over an elliptical shoal seated on a horizontal bed. Outside the shoal region the water depth is constant and equal to \( h = 0.457 \) m. The depth varies from \( h = 0.457 \) m (the base of the scatterer) to \( h = 0.1522 \) m at top of the elliptical shoal \( (x_1 = 12.5 \) m, \( x_2 = 0) \). Various cases have been considered in the above experimental work, including non-breaking and breaking waves. In order to demonstrate the applicability of the present model, we consider here the case N5 corresponding to the propagation/scattering of a narrow directional wave system, in the presence of additional effects of a current generated by the breaking of the higher water waves passing over the elliptic shoal (see [45,46]).

In the above case the significant waveheight of the incident spectrum is of the order of \( H = 16 \) cm in front of the scatterer (and \( H = 19 \) cm at the wavemaker). Furthermore, the peak frequency of the incident wave spectrum is \( T_p = 1.3 \) s and the directional spreading parameter \( \sigma_m = 10^\circ \). Thus, the spectral wave components essentially lie in the directional interval \( -30^\circ < \theta < 30^\circ \). Due to significant breaking over and behind the scatterer, a mean flow is generated, as presented and discussed in Yoon et al. [46]. In the present work this breaking-induced current has been approximated as illustrated in Figs. 8 and 9, based on data from the latter reference, which were obtained by using the SHORECIRC model [47]. In this case the maximum current velocity is of the order of \( 0.3 \) m/s. As schematically presented in these figures, the breaking current contributes to defocusing of the convergence of wave behind the elliptic shoal, redirecting wave energy outwards from the centerline, and producing significant modification on the
refraction pattern and the spatial distribution of wave energy, in comparison with corresponding results obtained without considering the current.

More specifically, in Figs. 8 and 9, snapshots of the real part of the calculated wave field \( \psi(x, z) \) on the surface are shown, normalized with respect to the amplitude of the incident wave, in the case of monochromatic waves of period \( T = 1.3 \, \text{s} \) (corresponding to the peak period of the incident wave system) and for \( \theta = 0^\circ \) and \( \theta = 15^\circ \) incidence angles, respectively. Numerical results with and without the effects of breaking-induced current have been obtained by the present model using 5 totally modes \((n = -1, 0, 1, 2, 3)\) and discretizing the CMS using second-order finite differences, and \(201 \times 201\) horizontal grid resolution, which corresponds to about 15 gridpoints per wavelength and was found enough for numerical convergence. We note here that the results presented in the left-hand side of the above figures are artificial, since they contain only the effects of the breaking-induced current over the variable bathymetry and not the dissipation of wave energy due to breaking. We are able to observe in the right subplots of Figs. 8 and 9 that without the effects of breaking-current (which is the case of small-amplitude waves) strong focusing of wave energy is observed a distance about 5 m downwave the scatterer. However, if breaking-induced current is also considered, we observe in the left-subplots that, except of the additional refraction effects, the current contributes to considerably spreading of the area of wave energy concentration (due to focusing by the elliptic shoal), which extends several meters in the downwave direction from the scatterer.

The above analysis permits us to calculate the complex wave potential \( \psi(x, z) \) for each single frequency \( \omega \) and direction \( \theta \) in an incident spectrum, as provided by the solution of the present CMS. Thus, the complex amplitude of other physical quantities of interest in the domain can be calculated, as for example, the free-surface elevation (obtained through Eq. (5.1b)), the horizontal and vertical wave velocity components, the pressure, etc. Then, on the basis of linear system theory applied to distributed systems, the transfer function associated with any physical quantity at any
point in the domain is obtained and the spatial distribution of the frequency spectrum over the domain can be calculated by spectral synthesis. First results in this direction have been presented in Belibassakis et al. [48], where the coupled-mode formulation is further extended to include dissipation effects due to bottom friction and wave breaking, using empirical formulae to determine the corresponding coefficients. In concluding this section, we show in Fig. 10, as a first example of the above approach, the calculated distribution of significant wave height along transect 4 in the N5 case of Vincent and Briggs [42] experiment. This transect is located normal to the centerline at a distance 6.1 m downwave the top of the elliptic shoal. In Fig. 10 our predictions are plotted against numerical results by Yoon et al. [46] and experimental data from Ref. [45], respectively. Present method results are indicated by a thick solid line and have been obtained by spectral synthesis, using 21 frequencies and 21 directions to discretize the incident directional wave spectrum. We notice in this figure that our results are in relatively good agreement with other methods and measured data, although they present a tendency to overestimate the wave height in the centerline, which is attributed to the specific expressions used for the dissipation coefficient.

7. Concluding remarks

A non-linear coupled-mode model is presented for the wave–current–seabed interaction problem, with application to wave scattering by steady, inhomogeneous currents in variable bathymetry regions. The vertical distribution of the scattered wave potential is represented by a series of local vertical modes containing the propagating mode and all evanescent modes, plus additional terms accounting for the satisfaction of the free-surface and bottom boundary conditions. Using the above representation, in conjunction with Luke’s variational principle, the wave–current–seabed interaction problem is reduced to a non-linear coupled system of differential equations on the horizontal plane. By linearising the above system an extension of the coupled-mode system previously developed by the authors is derived, for treating the additional scattering effects due to the presence of inhomogeneous currents in variable bathymetry regions. By keeping only the propagating mode in the vertical expansion of the wave potential, the present system reduces to an one-equation model, the modified mild-slope equation for wave scattering by depth and currents. The latter is shown to be compatible with Kirby’s mild-slope model concerning wave–current interaction over slowly varying topography, containing extra term carrying out additional effects of first and second horizontal derivatives of depth and current, and reducing exactly to the modified-mild slope equation in the case of no current. The present coupled-mode system is discretized on the horizontal plane by using second-order finite differences and numerically solved by iterations. Results are shown for various test cases demonstrating the usefulness of the present model, as well as the importance of the first evanescent modes and the additional sloping-bottom mode when the bottom slope is not negligible. The analytical structure of the present model facilitates its extension to fully non-linear waves, and to wave scattering by currents with more general structure. Finally, the extension of the present method to the transformation of wave incident systems (described by directional spectra) over variable bathymetry regions, including the effects of currents and dissipation due to bottom friction and breaking, will be studied in detail in another work.

Appendix A. Proof of the variational equation

The variation of the functional (3.1) is obtained as the sum of variations of all terms,

\[ \delta F = \delta \eta F + \delta \varphi F. \]  

(A.1)

The variation \( \delta \eta F \) is easily obtained as follows:

\[ \delta \eta F = - \rho \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} dt \, dx_1 \, dx_2 \times \left\{ \left( \frac{\partial \varphi}{\partial t} + U \frac{\partial \varphi}{\partial x_1} + g \eta \right) \delta \eta \right\} \]  

(A.2)

The first variation \( \delta \varphi F \) is calculated as follows (see also [49])

\[ \delta \varphi F = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} dt \, dx_1 \, dx_2 \int_{z=\eta(x,t)}^{z=h(x)} \left( \frac{\partial \varphi}{\partial t} + \left( U_1 + \frac{\partial \varphi}{\partial x_1} \right) \frac{\partial \varphi}{\partial x_1} + \left( U_2 + \frac{\partial \varphi}{\partial x_2} \right) \frac{\partial \varphi}{\partial x_2} + W + \frac{\partial \varphi}{\partial z} \right) \delta \varphi \, dz. \]  

(A.3)

Choosing the variation \( \delta \varphi \) to vanish at infinity \( (x_1, x_2) \to \pm \infty \) and at the ends of the time interval \( (t = t_1, t_2) \), and changing the order of differentiation and integration, we obtain for the first term in the right-hand side of Eq. (A.3),

\[ \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \int_{z=\eta(x,t)}^{z=h(x)} \frac{\partial \varphi}{\partial t} \delta \varphi \, dz \]

\[ = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \int_{z=\eta(x,t)}^{z=h(x)} \frac{\partial \varphi}{\partial t} \delta \varphi \, dz = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \frac{\partial \varphi}{\partial t} \delta \varphi \, dz. \]  

(A.4a)

where the integral involving the time derivative term (first term in the brackets) is dropped because the variation \( \delta \varphi \) vanishes at the ends of the time interval. For the second term in the right-hand side of Eq. (A.3), using integration by parts, we obtain

\[ \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \int_{z=\eta(x,t)}^{z=h(x)} \left( \frac{\partial \varphi}{\partial x_1} \right) \delta \varphi \, dz = \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \left( \frac{\partial}{\partial x_1} \int_{z=\eta(x,t)}^{z=h(x)} \delta \varphi \, dz \right) \delta \varphi \, dx_1 \]

\[ \times \left\{ \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \int_{z=\eta(x,t)}^{z=h(x)} \left( \frac{\partial \varphi}{\partial x_1} \right) \delta \varphi \, dz - \int_{t_1}^{t_2} \int_{s_1}^{s_2} \int_{x_1}^{x_2} \int_{z=\eta(x,t)}^{z=h(x)} \left( \frac{\partial \varphi}{\partial x_1} \right) \delta \varphi \, dx_1 \right\} \delta \varphi \, dx_1. \]  

(A.4b)
Similarly, the spatial derivative term (first term in the brackets) is dropped because the variation $\Delta \varphi$ vanishes at the ends of the interval. A corresponding result is also obtained for the third term in the right-hand side of Eq. (A.3). Finally, using integration by parts we obtain for the fourth term

$$
\int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left( \frac{\partial}{\partial z} \frac{\partial \varphi}{\partial z} \right) \frac{dx_2 \, dx_1 \, dy_2 \, dy_1 \, dz_2 \, dz_1}{z_2 - h(z_1)}
$$

Collecting together all terms, the variation $\Delta F$ finally becomes:

$$
\Delta F = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left( \frac{\partial}{\partial z} \frac{\partial \varphi}{\partial z} \right) \left( \left( U_1 + \frac{\partial}{\partial x} \frac{\partial \varphi}{\partial x} \right) \left( U_2 + \frac{\partial}{\partial x} \frac{\partial \varphi}{\partial x} \right) - \left( W + \frac{\partial \varphi}{\partial x} \right) \right) dz_2 \, dz_1 \, dy_2 \, dy_1 \, dx_2 \, dx_1
$$

Using Eqs. (2.1a) and (2.1c) in the above equation, and combining the result with Eqs. (A.1) and (A.2), we finally obtain the variational principle given by Eq. (3.4), modelling our hydrodynamical system.

References


