An enhanced coupled-mode system is developed for modelling wave-induced pressure and groundwater flow in variable bathymetry coastal regions, in the layer under the permeable seabed. The sea bottom is assumed of general shape, characterised by sloping parts and undulations, without restrictions concerning mildness of bottom slope and curvature. The present model is based on a simplified version of the storage and momentum equations, corresponding to large values of stiffness ratio of the porous medium, permitting prediction of wave-induced groundwater pressure and flow in the case when the elastic motion of the seabed is very small and could be approximately neglected. The representation of the wave pressure field is based on a local-mode series expansion in the water column and the porous medium, consisted of propagating and evanescent modes, and defined by formulating and solving vertical eigenvalue problems at each local horizontal position. The latter series is enhanced by appropriate terms (the sloping-interface mode) enabling consistent satisfaction of the interface conditions at the seabed and providing fast convergence. Numerical results are presented and compared against predictions by other methods and measured data, illustrating the applicability of the present model to variable bathymetry coastal regions and sandy beaches.
extension to treat wave propagation and dissipation in general 3D environments (Belibassakis et al., 2001) and to predict second-order and fully non-linear waves in variable bathymetry; see, e.g., Belibassakis and Athanassoulis (2002, 2011).

Useful information concerning the time-averaged groundwater circulation induced by wave set-up in a permeable sea bottom can be obtained by exploiting the solution of the slow-scale mean-flow equations in variable bathymetry regions, which are forced by the radiation stresses, the free-surface and the boundary stresses (see, e.g., Dingemans, 1997). Under the assumption that the groundwater flow is in the Darcy law regime, phase-averaged, porous flow velocities can be obtained in terms of the pressure gradient. In this case, the groundwater circulation is modelled by the Laplace’s equation, forced by the excess pressure on the sea bottom, which is induced by the mean flow (set-up); see e.g., Massel (2001). In previous works (Belibassakis and Athanassoulis, 2007) the above coupled-mode model has been further extended to include the effects of bottom friction and wave breaking, that are important for the more accurate calculation of the radiation stresses, especially on decreasing depth and in a beach region. Then, this model has been used, in conjunction with an iterative solver of the mean-flow equations, for the calculation of wave-induced set-up and mean flow in open and closed environments. Finally, the resulting phase-averaged mean-bottom pressure has been applied to obtain results concerning the induced groundwater circulation in a permeable beach, in the set-up region.

In the case of water wave propagation over a permeable seabed various dissipation mechanisms associated with the porous medium properties contribute in damping and modification of the wave characteristics. For seabed sediments consisting of coarse sand, the effects of percolation might become important. Moreover, in the case of finer grain sizes the sea bottom becomes deformable under wave loading and the effects of poro-elasticity are significant. The problem of the response of a poro-elastic bed to water waves has been analytically treated by Yamamoto et al. (1978) on the basis of Biot’s consolidation theory, in the case of infinite thickness of porous medium. The analysis has been further extended to the case of finite thickness by Hsu and Jeng (1994); see also Jeng and Hsu (1996), Lee et al. (2002) and Massel et al. (2005). Furthermore, effects of 3D bottom topography and sloping seabed have been studied in Zhang and Jeng (2005) and effects of breaking waves have been considered in Jeng and Zhang (2005). Comparison of existing poro-elastic models for wave damping in a porous seabed have been presented by Lin and Jeng (2003), and review of various solutions is provided in Jeng (2003). Moreover, the decay of progressive gravity waves propagating above a porous bed is studied in Williams and Jeng (2007), assuming potential flow in a two-layer system with a free surface and a sharp interface. In the case of seepage flows induced by long waves in unsaturated permeable seabed analytical solutions for pore-water pressure and velocity have been obtained by Liu et al. (2007), where also numerical solutions are obtained by simultaneously solving the Navier–Stokes equations for water wave motions and the diffusion equation in the seabed. More recently, a Navier–Stokes type model for porous flow is presented by Huang et al. (2011) and applied to simulate the propagation of water waves over a permeable rippled bed, and results are compared against predictions by the mild-slope model of Mase et al. (1995), which extends Kirby’s (1986) theory to waves propagating over permeable rippled beds.

Focusing on the region of water depths greater than the wave-breaking depth, in the present work a coupled-mode model is developed and applied to water wave propagation in variable bathymetry regions and its interaction with porous flow in the layer under the permeable seabed. The present model is based on a simplified version of the storage and momentum equations (see, e.g., Mei, 1983, Section13.2.4), corresponding to large values of stiffness ratio of the porous medium, permitting prediction of wave-induced groundwater pressure and flow in the case when the elastic motion of the seabed is very small and could be approximately neglected. On the other hand, the sea bottom topography is assumed to be of general shape, characterised by sloping parts and undulations, without mildness assumptions and restrictions concerning of its slope and curvature. The representation of the wave pressure field is based on a local-mode series expansion in the water column and the porous medium, consisted of propagating and evanescent modes, which are defined by formulating and solving vertical eigenvalue problems at each local horizontal position. The latter series is enhanced by appropriate terms (the sloping-interface mode) enabling consistent satisfaction of the interface conditions at the seabed and providing fast convergence. Using the above representation in a variational principle, a coupled-mode of horizontal equations is derived for the determination of the unknown pressure amplitudes. The present method extends previous analysis by Athanassoulis and Belibassakis (1999) for waves propagating in variable bathymetry regions to the permeable bottom case. Numerical results are presented and compared against predictions by other methods and measured data, illustrating the applicability of the present model to general bottom topography coastal regions and sandy beaches.

2. Formulation of the problem

The marine environment consists of a water layer bounded above by the free surface and below by the sea bottom, separating water from porous medium, which is terminated by a horizontally flat, impermeable bottom boundary. The variable water depth is denoted by $h$ and thus, the thickness of the permeable layer is $h_B = h$; see Fig.1. The porous medium is considered to be isotropic and homogeneous. The coefficient of permeability is denoted by $K = K_f / g$, where $K_f$ the coefficient of filtration and $g = \rho g$ the specific weight of water, $g$ is the acceleration due to gravity, and the porosity of the sandy bottom is denoted by $n_w$.

For simplicity we will restrict ourselves to a 2D problem, however the present theory can be extended to general 3D environments and obliquely incident waves; see Belibassakis et al. (2001) and Gerostathis et al. (2008) as concerns the treatment of 3D effects associated with the water-wave part in a general bottom topography terminated by a rigid impermeable seabed. It is assumed that the bottom surface exhibits a general variation, i.e. the bathymetry $h(x)$ is characterised by parallel, straight bottom contours lying between two regions of different depth: the deeper water region or region of incidence ($x < a$,...
where the depth is constant and equal to \( h = h_1 \) and the shallow water region \( (x > b, \) where the depth is constant and equal to \( h = h_1 \). A Cartesian coordinate system is introduced, with its origin at some point on the mean water level, and the z-axis pointing upwards.

When water waves propagate over a porous bed, as in the case of a sandy beach environment, fluid flow is induced to the porous medium which is forced to deform, and the bed response is in general a combination of fluid and solid mechanical effects. In the case when the stiffness of the soil is much larger than the pore fluid, the present formulation is based on a simplified version of the poroelastic model originally developed by Biot (see, e.g., Verruijt, 1969; Mei, 1983). According to the latter model, the linearised equations for water and soil skeleton, consist of the storage equation (see Mei, 1983, Section 13.2.4)

\[
n_w \nabla v_w + (1 - n_w) \nabla v_s = - \frac{n_w}{E^v} \frac{\partial p}{\partial t}.
\]  

expressing mass conservation in the porous medium, in conjunction with the momentum equations for the soil and the fluid. In Eq. (1) \( v_w = (u,w) \) and \( p \) denote the dynamic fluid velocity and pore pressure, respectively, and \( v_s \) is the velocity of deformation of the soil matrix. The momentum equations combined with Hooke’s law are finally put in the form (see Mei, 1983, Section 13, Eq. (2.47))

\[
G \left( \nabla^2 v_w + \frac{1}{1 - 2v} \nabla \nabla v_s \right) - \nabla p = n_w \rho_w \frac{\partial^2 v_w}{\partial t^2} + (1 - n_w) \rho_s \frac{\partial^2 v_s}{\partial t^2},
\]

where \( G \) is the shear modulus, \( v \) is Poisson’s ratio and \( \rho_s \) the density of the solid skeleton. The above equations involve the perturbation components of the dynamic velocity and pressure field in the porous medium, generated by the water wave motion, defined as the difference between the total field and the background field, which is the existing field in the absence of waterwave propagation. As described in Mei (1983, Sections 13.2.3 and 13.2.4) these equations are obtained by subtracting parts the momentum equations for the pore water and the soil matrix, written separately for the total field and the background field, and linearising. In the case of a flat seabed the background field is a static one (zero flow). In the case of variable bathymetry, the background field is assumed to be steady, possibly corresponding to slow seepage flow. The gravity body forces are present both in the momentum equations of the total field and in the momentum equations of the background field. Thus, after subtraction by parts and linearization, the body forces are cancelled at the leading order and the remainder in the momentum equations is very small and can be approximately neglected.

Furthermore, the momentum equations combined with the storage equation yield the following equation (see Mei, 1983, Section 13.2.4, Eq. (2.48))

\[
\nabla^2 p - \frac{n_w}{K} \frac{\partial p}{\partial t} + \rho \frac{\partial^2 v_w}{\partial t^2} = 0.
\]

where \( E \) in the above equations is the effective bulk modulus of elasticity of the fluid in the porous medium, taking into account possible content of gas (air) in the fluid. If the pore water is absolutely air-free, \( E \) in the above equations is the bulk modulus of elasticity of the water, taken \( E \approx 1.9 \times 10^9 \text{ Nm}^{-2} \). However, if the pore-water contains even small amounts of gas the bulk modulus drastically decreases as described by the following relation (see, e.g., Verruijt, 1969; Yamamoto et al., 1978):

\[
\frac{1}{E} = \frac{1}{E_0} + \frac{1 - S}{E_0}.
\]

where \( 1 - S \) is the air content and \( E_0 \) denotes the absolute pore-water pressure. The latter quantity, in the first layer(s) below the seabed, could be approximated by the atmospheric plus hydrostatic pressure due to the water column above the sea bottom. In shallow water, due to possible wave breaking and the entrance of air into the porous medium, as well as production of gases by the organisms living in the sand, \( 1 - S \) could be non-negligible and the apparent bulk modulus could be significantly different than the one for pure water. For example, using the above formula with 5% air content and atmospheric mean pressure (zero depth) the apparent modulus is calculated to be \( E = 2 \times 10^6 \text{ Nm}^{-2} \), that is 1000 times smaller than the corresponding one for pure water. Considering the soil skeleton of the porous medium material to be sand, with a representative shear modulus of the order \( G \approx 10^8 \text{ Nm}^{-2} \), the values of stiffness ratio \( G/E \) expressing the significance of pore-flow effects in comparison with elastic deformation effects, for various values of air content and water depths are shown in Fig. 2. We see in this figure that even for very small values of air content and low or moderate depths (indicated by a dashed line) the ratio \( G/E \) becomes high of the order of 10 and greater. Similar conclusions are derived in the case of other materials like dense soil or sandstone (using, e.g., the data in Mei, 1983, Section 13, Table 3.1). In this case, the elastic deformation effects are very small and Eqs. (1–3) could be simplified by disregarding the terms containing the velocities of the solid skeleton considering them to be much smaller that the corresponding ones associated with the pore fluid (i.e., \( |v_s| \ll |v_w| \)), obtaining the approximate relations

\[
\nabla^2 p - \frac{n_w}{K} \frac{\partial p}{\partial t} + \rho \frac{\partial^2 v_w}{\partial t^2} = 0
\]

Subsequently, we may use the second relation above to eliminate the divergence of pore-flow velocity in the first one, and we finally obtain the following equation:

\[
\nabla^2 p - \frac{n_w}{K} \frac{\partial p}{\partial t} \frac{\partial^2 v_w}{\partial t^2} = 0,
\]

that could be employed to describe wave propagation of pressure field in the porous medium considered. Moreover, induced groundwater flow velocity can be obtained from pressure gradient using the momentum equation of the fluid (Mei, 1983, Section 13, Eq. (2.45)), which in the present simplified case reads as follows:

\[
\rho \frac{\partial^2 v_w}{\partial t^2} + \frac{n_w}{K} \nabla v_w = - \nabla p.
\]

We note here that Eq. (6) without the last term in the left-hand side, which then becomes exclusively of diffusion type, is also
derived by applying the Darcy law, in conjunction with conservation of mass equation; see e.g., Bear (1972) and Liu et al. (2007, Eq. (2.2)). The same result has also been discussed by Yamamoto et al. (1978, Section 3.4) and Massel et al. (2005, Eq. (59)), in the case of soil of large stiffness saturated with a mixture of liquid and gas, based on a poro-elastic model.

The field Eq. (6) for monochromatic waves\( p(x,z,t) = \text{Re}[p(x,z)\exp(-i\omega t)] \), of angular frequency \( \omega \), reduces to the Helmholtz equation for the complex amplitude of the pressure\( p(x,z) \), in the vertical plane
\[
\frac{\partial^2 p}{\partial z^2} + \frac{\partial^2 p}{\partial x^2} + \psi^2 \rho p = 0, \quad -h_{b} < z < -h(x),
\]
(8)
characterised by the complex-valued wavenumber parameter
\[
\psi^2 = \frac{\rho_0 \omega^2}{E} + \left( \frac{h_{b}}{E} \right),
\]
(9)
where \( i = \sqrt{-1} \) and the imaginary part of \( \psi \) indicates the dissipative nature of the present flow model in the porous medium. As it will be illustrated in the sequel, diffusion effects are still dominant in the present model, which in addition admits solutions describing the propagation of coupled pressure waves in the porous and water media.

The wave field in the region \( D \) is excited by a harmonic incident wave of small amplitude, with direction of propagation normal to the depth-contours. In the framework of linearised wave water equations (see, e.g., Mei, 1983), the fluid motion is described by the complex 2D potential\( \phi(x,z) \), which is connected to the water-wave pressure through Bernoulli’s equation
\[
\phi(x,z) = -\frac{i}{\omega \rho_0} p(x,z), \quad \Phi(x,z,t) = \text{Re}[\phi(x,z,\mu) \exp(-i\omega t)],
\]
(10)
where \( \mu = \omega^2 / g \) denotes the frequency parameter, and the free-surface elevation is obtained in terms of the wave potential on the free surface as follows:
\[
\eta(x,t) = -\frac{1}{g} \frac{\partial \Phi(x,z=0,t)}{\partial t} = -\frac{1}{\rho g} \text{Re}[p(x,z=0) \exp(-i\omega t)].
\]
(11)
In the water region, the complex wave potential and thus also the pressure satisfy the Laplace equation in the vertical plane
\[
\frac{\partial^2 p(x,z)}{\partial z^2} + \frac{\partial^2 p(x,z)}{\partial x^2} = 0, \quad -h(x) < z < 0,
\]
(12)
supplemented by the free-surface boundary condition,
\[
\frac{\partial p(x,z)}{\partial z} - \mu p(x,z) = 0, \quad z = 0.
\]
(13)
Using Eq. (10) in Eq. (7), the wave flow velocities (complex amplitudes) in the water region and in the porous medium, respectively, are obtained from gradient of pressure as follows:
\[
v = -\frac{i}{\omega \rho_0} \nabla p, \quad -h(x) < z < 0, \quad v_w = -\frac{i}{\omega \rho_0 (1+iT)} \nabla p, \quad -h_{b} < z < -h(x),
\]
(14)
where \( T = n_{wp}/K_{wp} \). At the seabed surface \( z = -h(x) \), the vertical effective stresses and shear stress should vanish, and the fluid pressure is transmitted continuously from the sea to the pores in the seabed, Yamamoto et al. (1978), Hsu and Jeng (1994), and Jeng (2001). Furthermore, the fluid flux in the normal direction \( v \) across the seabed (see Fig. 1) should be conserved, i.e. \( (v - n_{wp}v) \cdot \mathbf{n} = 0 \); see, e.g., Mei (1983), Massel et al. (2005). Under the present assumptions and simplifications, the above conditions reduce to the requirement of continuity of pressure at the water-seabed interface
\[
p(x,z = -h(x) + 0) = p(x,z = -h(x) - 0),
\]
(15a)
and the continuity of normal water flux, which using Eq. (14) takes the form
\[
\frac{\partial p(x,z = -h(x) - 0)}{\partial v} = (z + i\beta) \frac{\partial p(x,z = -h(x) - 0)}{\partial v},
\]
(15b)
where the parameter \( z + i\beta = n_{wp}(1 - i\mu / 1 + T^2) \), \( \tau = n_{wp}/K_{wp} \), and the normal derivative on the interface \( z = -h(x) \) is defined as
\[
\frac{\partial}{\partial z} = \left[ 1 + \left( \frac{\partial h}{\partial x} \right)^2 \right]^{-1/2} \left( \frac{\partial}{\partial z} - \frac{\partial h}{\partial x} \frac{\partial}{\partial x} \right)
\]

Finally, on the flat horizontal impermeable boundary, \( z = -h_{b} \), a homogeneous Neumann condition applies
\[
\frac{\partial p(x,z = -h_{b})}{\partial n} = 0,
\]
(16)
ensuring zero normal flux. The studied problem in the present work consists of the field Eqs. (8) and (12) in the porous medium and water region, respectively, subject to the free surface boundary condition (13), the interface conditions (15a) and (15b) and the condition on the impermeable bottom (16). The field is excited by an incident harmonic wave propagating in the positive \( x \)-direction, as shown in Fig. 1. Due to the fact that \( \text{Im} \psi > 0 \), we expect that the solution of the coupled fluid wave and pore flow equations will exhibit decay in the direction of propagation. Thus, we shall employ the wave amplitude \( A \) associated with the incident wave field at the entrance section \( x = 0 \) of the variable bathymetry region in order to scale the solution of the present linear problem \( (p / \rho g A) \). Furthermore, in the case of low porosity \( (n_{wp} < 1) \) and thus, also \( |z + i\beta| < 1 \), Eq. (15b) reduces to a homogeneous Neumann boundary condition on the sea bottom, \( z = -h(x) \), and an uncoupled model is obtained (see also Massel et al., 2005). In this case, the water-wave part, Eqs. (12), (13) and (15b) can be independently solved permitting the calculation of the pressure on the sea bottom \( p(x,z = -h(x)) \). Subsequently, the porous pressure and flow wave field is obtained as a solution of Eqs. (8) and (16), forced by the water wave pressure on the sea bottom, Eq. (15a). Such an approach, based on a boundary integral formulation of the decoupled problem, has been recently presented in Belibassakis (2011).

3. Representation in the constant depth strips

The liquid domain \( D \) is decomposed in three parts \( D^{(m)}, \quad m = 1,2,3 \), where \( D^{(2)} \) is the variable bathymetry subdomain \( (a < x < b) \), \( D^{(1)} \) is the constant-depth region of incidence \( (x < a, h = h_2) \) and \( D^{(3)} \) the constant-depth region of transmission \( (x > b, h = h_3) \); see Fig. 1. The free-surface \( (z = 0) \), sea bottom \( (z = -h) \) and impermeable bottom \( (z = -h_{b}) \) boundaries are similarly decomposed.

Normal-mode type representations of the wave field in the regions of incidence and transmission \( D^{(m)}, m = 1,3 \), respectively, can be derived by separation of variables, in the following form concerning the scaled pressure field
\[
p(x,z) = \exp(ik_{n}^{(1)}x)A_{n}^{(1)}\exp(ik_{n}^{(3)}x) + \sum_{n=1}^{\infty} A_{n}^{(3)}\exp(-ik_{n}^{(1)}x)Z_{n}^{(1)}(z), \quad \text{for} \quad x < a
\]
(17a)
\[
p(x,z) = A_{n}^{(3)}\exp(ik_{n}^{(3)}x) + \sum_{n=1}^{\infty} A_{n}^{(3)}\exp(ik_{n}^{(3)}x)Z_{n}^{(3)}(z), \quad \text{for} \quad x > b.
\]
(17b)
The first term in Eq. (17a) represents the incident wave, the second the reflected one, and the rest terms are evanescent modes in \( D^{(1)} \). Also, the first term in Eq. (17b) is the transmitted wave and the rest terms are evanescent modes in \( D^{(3)} \). The functions \( Z_{n}^{(1)}(z) \) controlling the vertical structure of each mode,
and the numbers \( k_n^{(m)} \), \( m = 1,3 \), in the above equations, are obtained as the eigenfunctions of the following, vertical eigenvalue problems:

\[
\frac{d^2 z_n^{(m)}}{dz^2} + \left( k_n^{(m)} \right)^2 z_n^{(m)} = 0, \quad -h_g < z < h_m \text{ and } -h_m < z < 0, \\
(18a)
\]

\[
\frac{d z_n^{(m)}}{dz} - \mu z_n^{(m)} = 0, \quad z = 0, \quad \text{and} \quad \frac{d^2 z_n^{(m)}}{dz^2} = 0, \quad z = -h_g, \\
(18b, c)
\]

\[
Z_n^{(m)}(z = -h_m + 0) = Z_n^{(m)}(z = -h_m - 0), \quad z = -h_m, \\
(18d)
\]

\[
\frac{d Z_n^{(m)}}{dz}(z = -h_m + 0) = (\alpha + i\beta) \frac{d Z_n^{(m)}}{dz}(z = -h_m - 0), \quad z = -h_m. \\
(18e)
\]

where the wavenumber parameter \( k \) appearing in Eq. (18a) is defined as

\[
k = 0, \text{ in } -h_m < z < 0, \quad \text{and} \quad k = \psi \text{ in } -h_g < z < -h_m. \quad (19)
\]

The solution of Eqs. (18a)–(18e) generates vertical bases in \( D^{(m)}, m = 1,3 \), respectively; see, e.g., CODDINGTON and LEVINSON (1955) and Tichmarsh (1962). The numbers \( k_n^{(m)}, n = 0,1,2,3, \ldots \), are obtained as the \( \kappa \)-roots of the following dispersion equation, formulated at the water depths \( h_m, m = 1,3 \),

\[
-\kappa \sinh(k h_m) + \mu \cosh(k h_m) = -(\alpha + i\beta) \frac{\sinh(k h_g - h_m)}{\cosh(k h_g - h_m)},
\]

\[
\times \left( \cosh(k h_m) - \frac{\mu}{k} \sinh(k h_m) \right),
\]

\[
(20)
\]

where \( \iota = \sqrt{\psi^2 - k^2} \). The corresponding eigenfunctions are given by

\[
Z_n(z) = \cosh(\kappa z) + \frac{\kappa}{k} \sinh(\kappa z), \quad -h_m < z < 0, \quad -h_g < z < -h_m.
\]

\[
(21a)
\]

\[
Z_n(z) = \cosh(\iota(z + h_g)) - (\mu/k) \frac{\sinh(k h_m - \iota z)}{\cosh(k h_m - \iota z)}, \quad -h_g < z < -h_m.
\]

\[
(21b)
\]

To illustrate the distribution of roots of the above dispersion equation on the complex \( \kappa \)-plane and introduce the method of numerical calculation, consider first the simple case \( \alpha + i\beta = 0 \), corresponding to the uncoupled model (discussed at the end of the previous section). In this case, Eq. (20) reduces to the water-wave dispersion relation in constant depth, and the roots are distributed as shown in Fig. 3 using circles.

![Fig. 3. Distribution of the roots of Eq. (20) on the complex \( \kappa \)-plane.](image)

For relatively small \( n_0 \) and \( |\alpha + i\beta| \), the roots of Eq. (20) can be calculated iteratively, following a perturbation approach, starting from an initial estimation corresponding to \( n_0 = \alpha + i\beta = 0 \). Then, on the basis of the previous estimation the following parameter(s) are calculated:

\[
\gamma = -(\alpha + i\beta) \frac{\sin(\iota h_g - h_m)}{\cosh(\iota h_g - h_m)}.
\]

\[
m = 1,3, \text{ and a new estimation of } \kappa \text{-roots is obtained from Eq. (20), written in the form}
\]

\[
(-\kappa^2 + \mu^2) \sinh(k h_m) + (\mu^2 - k^2) \cosh(k h_m) = 0.
\]

(23)

Extensive numerical experience shows that, for \( |\alpha + i\beta| < 1 \), the above procedure converges, and the roots are transposed in the complex plane as depicted in Fig. 3 by using crosses. Also, the distribution of eigenvalues remains similar and no new spurious roots emerge. Thus, the wavenumber \( k_0 \), associated with the propagating mode, contains now a small and positive imaginary part, which represents dissipation effects in the propagation direction (here the positive x-direction). Similarly, the wavenumbers \( k_n, n \geq 1 \), are transposed from the imaginary axis to the first quadrant containing now small and positive real parts. The above remarks justify the forms provided by Eq. (17) concerning the general representations of the wave field in the constant depth regions of incidence and transmission, respectively.

As an example we consider here short waves of period \( T = 5 \text{ s} \) and wave height \( 2A = 0.3 \text{ m} \), and longer waves of period \( T = 8 \text{ s} \) and wave height \( 2A = 0.5 \text{ m} \), propagating in a water layer of constant depth \( h = 2 \text{ m} \) overlying a sandy bottom of constant thickness \( 2 \text{ m} \) (so that \( h_g = 4 \text{ m} \)). For presenting results and comparisons, we use here data from a series of experiments by Masell et al. (2005), where the permeability of the porous medium is \( K_f = 2.9 \times 10^{-4} \text{ m s}^{-1} \), porosity \( n_0 = 0.26 \) and the Young modulus \( E = 10^8 \text{ N m}^{-2} \). Moreover, high degree of saturation of water by air has been reported, so that in the examined case the apparent bulk modulus of pore water has been estimated to be \( E = 4.8 \times 10^5 \text{ N m}^{-2} \). Results concerning the initial estimation of the first non-dimensional wavenumbers \( \{k_0, h, n = 0,1,2,3\} \) and the corresponding final calculated values are presented in Table 1. In the same table the values of the coefficient \( \alpha + i\beta \) in the two cases considered are listed. From the wavenumber associated with the propagating mode \( n = 0 \) we see that, in constant depth, the shorter waves \( (T = 5 \text{ s}) \) attenuate by 1% over a distance of 5.1 wavelengths (which is estimated \( z = 20.95 \text{ m} \)), while in the case of longer waves \( (T = 8 \text{ s}) \) the same attenuation happens in 3.6 wavelengths (where \( z = 34.67 \text{ m} \), respectively).

The corresponding vertical eigenfunctions \( Z_n(z), n = 0,1,2,3 \), are comparatively plotted in Fig. 4. The real parts of these complex functions are shown by using solid lines and the imaginary parts by using dashed lines, respectively. We clearly observe the continuity of these functions at the interface \( z = h_g = 2 \text{ m} \). On the contrary, the vertical derivative of these functions exhibits there a noticeable discontinuity, as anticipated by Eq. (18e). We also observe in this figure that the essential support of the imaginary parts of the eigenfunctions is in the

| Table 1 |

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<thead>
<tr>
<th>( k_0h )-initial ( T = 5 \text{ s} )</th>
<th>( k_0h )-final ( T = 5 \text{ s} )</th>
<th>( k_0h )-initial ( T = 8 \text{ s} )</th>
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porous layer, and that the longer waves \((T=8 \text{ s})\) penetrate deeper in the sandy bottom than the shorter ones \((T=5 \text{ s})\).

In the same example, a comparison concerning the vertical distribution of dynamic pore-pressure between theoretical predictions based on the present model (lines) and experimental data (circles, from Massel et al. (2005)) is presented in Fig. 5 for the shorter waves \((T=5 \text{ s})\) and in Fig. 6 for the longer waves \((T=8 \text{ s})\), respectively. The numerical results presented in these
problem in the bounded subdomain $D^{(2)}$, satisfying Eqs. (8) and (12), the boundary conditions (13) and (16), on the free surface $\partial D_F (z=0)$ and the impenetrable bottom $\partial D_B (z=-h_0)$, respectively, the interface conditions (15a) and (15b) on $\partial D_I (z=0)$, and the following matching conditions:

\[
p^{(2)}(x,z) = p^{(1)}(x,z), \quad \frac{\partial p^{(2)}}{\partial x} = \frac{\partial p^{(1)}}{\partial x},
\]

\[
x = a, \quad -h_0 < z < -h_1 \quad \text{and} \quad -h_1 < z < 0,
\]

\[
p^{(2)}(x,z) = p^{(3)}(x,z), \quad \frac{\partial p^{(2)}}{\partial x} = \frac{\partial p^{(3)}}{\partial x},
\]

\[
x = b, \quad -h_0 < z < -h_1 \quad \text{and} \quad -h_3 < z < 0,
\]

on the vertical interfaces $\partial D_I^{(12)} (x=a)$ and $\partial D_I^{(23)} (x=b)$, shown by using dashed lines in Fig. 1. For simplicity, in the sequel the pressure field $p^{(2)}(x,z)$ in the variable bathymetry subdomain $D^{(2)}$ is denoted as $p(x,z)$. In order to state a variational formulation of the transmission problem we consider the functional

\[
F[p,A_1^{(1)}, A_3^{(3)}] = \frac{1}{2} \int_{D_0} (\nabla p)^2 \, dV + \frac{\lambda + i\eta}{2} \int_{D_0} \left[ (\nabla p)^T \cdot e \cdot \nabla p \right] \, dV - \frac{1}{2} \mu \int_{D_0} p^2 \, dS + f,
\]

where $D_0 = \{-h(x) < z < 0, \ a < x < b \}$, $D_1 = \{-h_0 < z < -h(x), \ a < x < b \}$ are the water and porous layers, respectively, and $f$ consists of boundary integral terms on the vertical interfaces $\partial D_I^{(12)} (x=a)$ and $\partial D_I^{(23)} (x=b)$, enabling the satisfaction of the matching conditions Eqs. (24); see also Anthanassoulis and Belibassakis (1999). The arguments of functional are the continuously distributed values of the function $p(x,z)$, over the field points in $D^{(2)}$, and the sets of coefficients $\{A_1^{(1)}, \ n=0,1,2,\ldots\}$ and $\{A_3^{(3)}, \ n=0,1,2,\ldots\}$, appearing in the representations of $p^{(1)}$ and $p^{(3)}$, respectively. The variational formulation of the transmission problem is stated as

\[
\delta F[p,A_1^{(1)}, A_3^{(3)}] = 0.
\]

Indeed, by calculating the first variation of functional (25) and using Green’s theorem, the variational Eq. (26) takes finally the form

\[
- \int_{D_1} \Delta p \delta p \, dV - (\lambda + i\eta) \int_{D_1} (\Delta p + \psi^2 p) \delta p \, dV + \int_{D_1} \left( \frac{\partial p}{\partial x} \right)^2 \delta p \, dS + (\lambda + i\eta) \int_{D_1} \left( \frac{\partial p}{\partial x} \right)^2 \delta p \, dS + \int_{\partial D_1^{(12)}} \chi(p-p^{(1)}) \delta \frac{\partial p^{(1)}}{\partial x} \, dS - \int_{\partial D_1^{(23)}} \chi(p-p^{(3)}) \delta \frac{\partial p^{(3)}}{\partial x} \, dS - \int_{\partial D_1^{(12)}} \chi \left( \frac{\partial p^{(1)}}{\partial x} \right) \delta p \, dS - \int_{\partial D_1^{(23)}} \chi \left( \frac{\partial p^{(3)}}{\partial x} \right) \delta p \, dS = 0.
\]

where $\chi = x + i\eta$ for points in the lower (porous) layer and $\chi = 1$ in the upper (water) layer, respectively. The usefulness of the above variational principle hinges on the fact that it leaves us the freedom to choose any particular representation for the unknown field $p(x,z) \in D^{(2)}$. One possible choice, facilitating the treatment of studied wave problem in general domains, without restrictions concerning the bottom slope or curvature, will be presented in the following section.

4. Variational formulation

By exploiting the representations (17a) and (17b), the examined problem can be formulated as a transmission boundary value

5. The local-mode representation

The standard local-mode representation (briefly denoted by SR) of the pressure field $p$ in the intermediate subdomain $D^{(2)}$ is
given by

\[ p(x,z) = \sum_{n=0}^{\infty} P_n(x)Z_n(z;x). \]  

(28)

In the above expansion the functions \( Z_n(z;x) \), \( n=0,1,2,... \), are obtained as the eigenfunctions of the following local vertical eigenvalue problem (for any horizontal position \( x \) in the interval \( a < x < b \)):

\[ \frac{\partial^2 Z_n(z,x)}{\partial z^2} + (k^2 - \lambda_n(x))Z_n(z;x) = 0, \quad \text{in} \quad -h_0 < z < -h(x) \]

and \( -h(x) < z < 0 \),

\[ \frac{\partial Z_n(z)}{\partial z}(z=0;x) = 0, \quad \frac{\partial Z_n(z=-h_0;x)}{\partial z} = 0, \]  

(29a)

where \( k=0, \) in \( -h(x) < z < 0 \), and \( k=\psi, \) in \( -h_0 < z < -h(x) \), in conjunction with the interface conditions

\[ Z_n(z=-h(x)+0;x) = Z_n(z=-h(x)-0;x), \]  

(29d)

and \( \lambda_n(x), n=0,1,2,... \), are the corresponding eigenvalues. In order to emphasise the parametric dependence of the above problem on the horizontal position \( x \), we use the terms local eigenvalues and local eigenfunctions, for \( \lambda_n(x) \) and \( Z_n(z;x) \), respectively. Using Eqs. (29), for each horizontal \( x \)-position, the following orthogonality conditions can be derived (see, e.g., Boyles, 1984, Section 3.3)

\[ \langle Z_n, Z_m \rangle = \int_{-h(x)}^{0} Z_n(z)Z_m(z)dz + (\alpha+i\beta) \times \int_{-h(x)}^{-h_0} Z_n(z)Z_m(z)dz = s_n \delta_{nm}, \]  

(30)

where \( s_n \) are finite numbers and \( \delta_{nm} \) is Kronecker’s delta. Furthermore, using the smoothness properties of the depth function \( h(x) \) we easily obtain that \( Z_n(z;x) \) converge to \( Z_n^{(0)}(z) \) as \( x \to a \) and to \( Z_n^{(1)}(z) \) as \( x \to b \), respectively.

In Eq. (28), the first term \( (n=0) \), corresponds to the local propagating mode, and the rest terms \( (n \geq 1) \) to the evanescent modes. Also, \( P_n(x) \) are the coefficients of a spectral expansion of \( p(x,z) \) with respect to the local vertical basis \( \{ Z_n(z;x), n=0,1,2,... \} \). Extensive numerical evidence has shown that the (complex) mode amplitudes \( P_n(x) \) present the rate of decay \( |P_n(x)| = O(n^{-2}), \quad n \to \infty \), for each \( a < x < b \), which implies a slow rate of convergence of the series (28). This is due to the fact that, for any truncated version of these series, the approximate field

\[ p_N(x,z) = \sum_{n=0}^{N} P_n(x)Z_n(z;x), \quad N < \infty, \]  

(31)

satisfies at \( z = -h(x) \) the condition

\[ \frac{\partial p(x,-h(x)+0)}{\partial z} = (x+i\beta) \frac{\partial p(x,-h(x)-0)}{\partial z}, \quad a < x < b, \]  

(32)

which is incompatible with the interface condition Eq. (15b), when \( dh/dx \neq 0 \). Similarly as in the case of water waves propagating in variable bathymetry regions (Athanasoulis and Belibassakis, 1999), we introduce a specific continuous field element \( F(x,z) \), such that the difference

\[ p(x,z) = p(x,z) - F(x,z) \]  

(33)

satisfies exactly the same boundary and interface conditions as the vertical eigenfunctions \( Z_n(z;x) \), i.e.

\[ \frac{\partial p(x,z=0)}{\partial z} - \mu p(x,z=0) = 0, \quad \frac{\partial p(x,z=-h_0)}{\partial z} = 0. \]  

(34a, b)

Thus, the residual field \( p_N(x,z) \) is expanded in terms of the local basis \( \{ Z_n(z;x), n=0,1,2,... \} \), namely

\[ p_N(x,z) = p(x,z) - F(x,z) = \sum_{n=0}^{\infty} P_n(x)Z_n(z;x), \]  

(35)

where the latter series exhibiting much better convergence properties in the \( z \)-intervals \( -h_0 < z < -h(x) \) and \( -h(x) < z < 0 \). In fact, the series in the r.h.s. of Eq. (35) converges as fast as \( \sum_{n=1}^{\infty} n^{-n+1} \) and, thus, it can be legitimately differentiated term-by-term, leading to convergent expressions concerning the field derivatives.

We proceed now to construct an appropriate representation for the function \( F(x,z) \). A possible choice is in the form

\[ F(x,z) = P_{-1}(x)Z_{-1}(z;x), \]  

(36)

where

\[ P_{-1}(x) = \left[ p \right] = \left[ \frac{\partial p(x,z=0)+h(x)}{\partial z} - (x+i\beta) \frac{\partial p(x,z=0)}{\partial z} \right], \]  

(37)

and \( Z_{-1}(z;x) \) is any sufficiently smooth function in the intervals \( -h_0 < z < -h(x) \) and \( -h(x) < z < 0 \), satisfying the boundary conditions

\[ \frac{\partial Z_{-1}(z=0;x)}{\partial z} - \mu Z_{-1}(z=0;x) = 0, \quad \frac{\partial Z_{-1}(z=-h_0;x)}{\partial z} = 0, \]  

(38a, b)

and the interface conditions

\[ Z_{-1}(z=-h(x)-0;x) = Z_{-1}(z=-h(x)+0;x), \]  

(38c)

and

\[ \left[ Z_{-1} \right] = \left[ \frac{\partial Z_{-1}(z=-h(x)+0;x)}{\partial z} - (x+i\beta) \frac{\partial Z_{-1}(z=-h(x)-0;x)}{\partial z} \right] = 1, \]  

(38d)

for each horizontal position \( a < x < b \). A specific convenient form of the function \( Z_{-1}(z;x) \) is given by the polynomial

\[ Z_{-1}(z;x) = h(x) \left[ \left( \frac{z}{h_0} \right)^4 + \left( \frac{z}{h_0} \right)^3 \right], \quad \text{in} \quad -h(x) < z < 0, \]  

(39)

and \( Z_{-1}(z;x) = 0, \quad \text{in} \quad -h_0 < z < -h(x). \) However, other choices are also possible. Conditions (38a,b) implies that both \( F(x,z) \) and \( p_N(x,z) \) satisfy the free-surface boundary condition (13) and the impenetrable-bottom condition (16). Also, condition (38c) implies that both \( F(x,z) \) and \( p_N(x,z) \) satisfy the continuity condition (15a). Finally, the inhomogeneous condition (38d), in conjunction with Eqs. (36) and (37), implies that

\[ \left[ p_R \right] = \left[ p \right] - \left[ F \right] = 0. \]  

(40)

The function \( P_{-1}(x) \) can be interpreted as an additional degree of freedom, accounting for the non-homogeneity in the vertical derivative caused by the sloping interface. By substituting Eq. (36) into Eq. (35), we obtain the following, enhanced local-mode representation in the range dependent subdomain \( D^{(2)} \):

\[ p(x,z) = P_{-1}(x)Z_{-1}(z;x) + \sum_{n=0}^{\infty} P_n(x)Z_n(z;x) = \sum_{n=-1}^{\infty} P_n(x)Z_n(z;x). \]  

(41)

The additional term \( P_{-1}(x)Z_{-1}(z;x) \), included in the right-hand side of Eq. (41), will be called the sloping-interface mode. As it will be illustrated in Section 7 below, where numerical results are presented and discussed, an important consequence of the introduction of the additional sloping-interface mode in the enhanced local-mode series (41) is that its coefficients exhibit an improved
rate of decay:
\[ |P_n| = O(n^{-\delta}), \quad \text{as} \quad n \to \infty, \tag{42} \]
which, from the numerical point of view, implies that considerably fewer terms are required to be retained in order to obtain convergent results.

The concept of the additional mode has been introduced and studied in the context of small-amplitude water waves propagating over variable bathymetry regions in Athanassoulis and Belibassakis (1999). The improved convergence of the enhanced local-mode series facilitated applications in complex 3D domains (Belibassakis et al., 2001), and multichromatic wave systems (Gerostathis et al., 2008). This approach has been further extended to treat weakly and fully non-linear water waves in Belibassakis and Athanassoulis (2002, 2011), hydrodynamic interaction problems with floating elastic bodies (Belibassakis & Athanassoulis and Athanassoulis, 2006) and acoustic propagation in the sea (Athanassoulis et al., 2008).

6. The coupled-mode system

Assuming that \( p(x,z) \) is represented by means of the enhanced local-mode series (41), the functional \( F[p,A_n^{(1)},A_n^{(2)}] \) is transformed to an equivalent form of the form \( F(P_n^{(1)},A_n^{(2)}) \), implying that the degrees of freedom associated with the admissible wave field \( p(x,z) \) in \( D^2 \) are the modal amplitudes \( P_n^{(1)}, a < r < b \), \( n = -1,0,1,2, \ldots \). Associated with the vertical interfaces \( \partial D^{(2)} \) and \( \partial D^{(3)} \) are the degrees of freedom \( P_n^{(2)}, a = 0,1,2, \ldots \) and \( P_n^{(3)}, b = 0,1,2, \ldots \) of the amplitude values at the left-endpoint \( x = a \), and at the right-endpoint \( x = b \), respectively, as well as the sets of coefficients \( \{ A_n^{(1)}, n = 0,1,2, \ldots \} \) and \( \{ A_n^{(2)}, n = 0,1,2, \ldots \} \). Especially for the sloping-bottom amplitude \( P_{-1}(x) \), the following end conditions are imposed
\[
P_{-1}(a) = P_{-1}(b) = 0 \quad \text{and} \quad dP_{-1}(a)/dx = dP_{-1}(b)/dx = 0, \tag{43} \]

in accordance with Eq. (37), in conjunction with boundary condition (15b), and the smoothness assumptions concerning the depth function \( h(x) \) at the points \( x = a \) and \( x = b \), respectively, where the interface \( \partial D \) becomes flat horizontal.

6.1. Derivation of the coupled-mode system

In order to derive the coupled-mode equations, we first note that by introducing the enhanced local-mode series (41) into the variational principle, Eq. (27), the third and the fourth terms in the right-hand side of Eq. (27) vanish, due to the properties (29b,c) and (38a,b) of the vertical basis functions \( Z(x,z) \). Next, we assume that all the variations except \( \delta p(x,z) \) in \( D^{(2)} \) are kept zero. Thus, the last four integrals of the variational equation are dropped, obtaining
\[
- \int_{D_1} \Delta p \, \delta p \, dV - (x + i\beta) \int_{D_1} (\Delta p + \psi^2 p) \, \delta p \, dV + \int_{\partial D_1} \left( \psi p(x,z = -h(x) + 0) - (x + i\beta) \psi p(x,z = -h(x)-0) \right) \, \delta p \, dS = 0. \tag{44} \]

By using the enhanced representation for \( p(x,z) \), Eq. (41), and its consequence \( \delta p(x,z) = \sum_{n=-\infty}^{\infty} \delta P_n(z)Z(x,z) \) into Eq. (44), we obtain
\[
\sum_{m=-\infty}^{\infty} \int_{D_1} \left[ \frac{d^2 P_n(x,z)}{dx^2} + b_{mn}(x) \frac{dP_n(x,z)}{dx} \right] dx + c_{mn}(x) \frac{dP_n(x)}{dx} = 0, \tag{45} \]

where the \( x \)-dependent coefficients \( a_{mn}(x), b_{mn}(x), c_{mn}(x) \) are defined by
\[
a_{mn}(x) = \langle Z_n,Z_m \rangle, \tag{46a} \]

\[
b_{mn}(x) = 2 \left( \delta Z_n/\delta x \right) Z_m + \frac{dh(x)}{dx} \left[ Z_n \right] Z_m(z = -h), \tag{46b} \]
\[
c_{mn}(x) = \left( \delta^2 Z_n/\delta x^2 + \delta Z_n/\delta z \right) Z_m + \langle \delta Z_n,Z_m \rangle + \delta_{x=m} \delta_{z=0} Z_m(z = -h) + \frac{dh(x)}{dx} \left( \delta Z_n(-h(x)+0)/\delta x - (x + i\beta) \delta Z_n(-h(x)-0) \right) Z_m(z = -h). \tag{46c} \]

In the above equations, \( \delta_{x=m} \) is Kronecker’s delta,
\[
\left[ Z_n \right] = \varepsilon Z_n(z = -h(x)+0; x) \quad \varepsilon \frac{dh(x)}{dx} \left( -h(x)-0; x \right), \tag{47a} \]

and
\[
\langle f,g \rangle \equiv \int_z \left( \varepsilon f(x)g(z)dz + (x + i\beta) \int_z \frac{dh(x)}{dx} f(x)g(z)dz \right). \tag{47b} \]

Since \( \delta P_m(r) \), \( m = -1,0,1,2, \ldots \), are assumed arbitrary, independent variations, Eq. (45) is equivalent to the following coupled-mode system (CMS):
\[
\sum_{n=-\infty}^{\infty} a_{mn}(x) \frac{d^2 P_n(x)}{dx^2} + b_{mn}(x) \frac{dP_n(x)}{dx} + c_{mn}(x) P_n(x) = 0, \quad a < x < b, \quad m = -1,0,1, \ldots, \tag{48} \]

of second-order equations with respect to the unknown modal amplitudes \( P_n(x) \). We remark here that for porosity \( n_n \to 0 \), and thus also \( (x + i\beta) \to 0 \), the coefficients, Eqs. (46), and the above coupled mode system, Eq. (48), reduce to the ones derived by Athanassoulis and Belibassakis (1999) for the propagation of small amplitude water waves in variable bathymetry regions over an impermeable seabed.

6.2. Boundary conditions for the modal amplitude functions

Since the system (48) is equivalent to Eq. (45), retaining the former renders the latter an identity. Thus, the first four terms of the variational Eq. (27) can be dropped, and this equation is simplified to:
\[
+ \int_{\partial D^{(2)}} \chi(p-p^{(1)}) \frac{\partial p^{(1)}}{\partial x} dS - \int_{\partial D^{(2)}} \chi(p-p^{(2)}) \frac{\partial p^{(2)}}{\partial x} dS - \int_{\partial D^{(3)}} \chi(p^{(3)}) \frac{\partial p^{(3)}}{\partial x} dS = 0. \tag{49} \]

Assuming now that all variations, except \( \delta p \) on \( \partial D^{(2)} \), are kept zero, we obtain the following equation:
\[
\int_{z = -h(x)}^{z = 0} \left( \frac{\partial p^{(1)}}{\partial x} \right) dz + (x + i\beta) \int_{z = -h(x)}^{z = -h(y)} \left( \frac{\partial p^{(1)}}{\partial x} \right) dz = 0. \tag{50} \]

Expressing the horizontal derivative \( \partial p^{(1)}/\partial x \) of the pressure on the matching boundary \( \partial D^{(2)} \) by termwise differentiation of the series (17a), and using it in Eq. (50), in conjunction with the orthogonality conditions (30), we obtain after some algebra the following set of conditions at \( x = a \):
\[
\frac{dP_n(a)}{dx} = -ik^{(1)}_n \exp(ik^{(1)}_n a), \tag{51a} \]
\[
\frac{dP_n(a)}{dx} = -ik^{(1)}_n A^{(1)}_n \exp(-ik^{(1)}_n a), \quad n = 1,2, \ldots. \tag{51b} \]

Further, by considering that all variations in Eq. (49), except \( \partial (p^{(1)}/\partial x) \) on \( \partial D^{(2)} \), are zero, we obtain
\[
P_0(a) = \exp(i\beta_0) + A_0 \exp(-ik^{(1)}_n a), \tag{52a} \]
\[
P_n(a) = A_n \exp(-ik^{(1)}_n a), \quad n = 1,2, \ldots \tag{52b} \]
By eliminating the unknown coefficients $A_k^{(n)}$, $n = 0, 1, 2, \ldots$ in the above equations, we obtain the following boundary conditions concerning the modal amplitudes at $x = a$,

$$\frac{dP_0(a)}{dx} + ik_1^{(1)}P_0(a) = 2ik_0^{(1)} \exp(i k_0^{(1)} a),$$

$$\frac{dP_n(a)}{dx} + ik_n^{(1)}P_n(a) = 0, \quad n = 1, 2, \ldots$$

(53)

Following the same procedure in connection with the terms of Eq. (49) defined on the right matching boundary $D^{(2)}$, a similar set of boundary conditions is derived at the right end-point defined by Eq. (55). The free-surface elevation is plotted by using thick line (incident waveheight functions of $x$-position, for $n = 0, 1, 2, \ldots$).

7. Presentation of numerical results

The numerical solution of the present problem is obtained by first calculating the coefficients $a_{mn}(x)$, $b_{mn}(x)$, $c_{mn}(x)$, which are defined in terms of the local vertical basis $Z_0(2x)$, $n = 0, 1, 2, \ldots$ at each horizontal $x$-position. The solution of the vertical eigenvalue problem, for $n = 0, 1, 2, \ldots$, is expressed through equations similar to Eq. (21), and involves the local eigenvalues $\kappa_n(x)$, $n = 0, 1, 2, \ldots$.

The latter are calculated through the local dispersion equation of the form of Eq. (20), formulated at each horizontal $x$-position. Starting from the values corresponding to the impermeable case ($z + i\beta = 0$), a number of 10–20 iterations usually suffice to obtain the final set, for all $a < x < b$, as described in Section 2.

Subsequently, the discrete version of the present CMS (48) is obtained by truncating the local-mode series (41) to $N_n$, retaining thus a total number of $N_n = N_m + 2$ terms (modes), and using central, second-order finite differences to approximate the $x$-derivatives in the system. Discrete boundary conditions are obtained by using second-order forward and backward differences to approximate the horizontal derivatives in Eqs. (53), (54) and (43), at the ends $x = a$ and $x = b$. Thus, the discrete scheme is uniformly of second order in the horizontal direction. The coefficient matrix of the discrete system is block structured, with 3-diagonal blocks, having a total dimension $N_n(N_n + 1)$, where $N_n$ is the number of segments subdividing the interval $a \leq x \leq b$.

7.1. Case of a smooth shoal

In order to illustrate the effects of variable bathymetry (sloping bottom) on the hydrodynamic behaviour of the present system, we first examine the case of a smooth underwater shoal. The sea bottom profile (see Figs. 7 and 8), separating water from porous medium, is characterised by the following depth function:

$$h(x) = \frac{h_1 + h_3}{2} - \frac{h_1 - h_3}{2} \tanh(c(x - x_m),$$

$$a = 0 < x < b = 400 m, x_m = (a + b)/2,$$

(55)

Fig. 7. Pressure distribution ($p/\mu gA$) in the water and in the porous bottom layer in the case of short waves $T = 5$ s propagating over the smooth shoal with depth function defined by Eq. (55). The free-surface elevation is plotted by using thick line (incident waveheight $H = 0.3$ m).
Figs. 7 and 8, respectively. In these plots the real and imaginary parts of the non-dimensional dynamic pressure field \( p/p_0 A \) are illustrated using contour lines, where \( A \) denotes the amplitude of the incident wave at the entrance section \( (x=0 \text{ m}) \). In the same plots the calculated free-surface elevation is also shown by using thick lines. To obtain numerical results, in this example, a total number of \( N_1=8 \) \( (N_0=6) \) modes and \( N=100 \) segments subdividing the horizontal interval have been used, which was found enough for numerical convergence. We clearly observe in these figures that the wave field is continuous at the sea bed, exhibiting slope discontinuity in the normal direction, as anticipated by the interface conditions Eq. (15). The contour lines intersect the flat-horizontal impermeable bottom \( (z=-6 \text{ m}) \) perpendicularly, which is evidence of fulfillment of the bottom boundary condition, Eq. (16). This fact is also illustrated in Fig. 9, where the corresponding vertical distributions of the wave pressure field at three horizontal positions, at the entrance \( (x=0 \text{ m}) \), the middle \( (x=20 \text{ m}) \), and the exit \( (x=40 \text{ m}) \) of the above inhomogeneous waveguide are shown, in the case of shorter incident waves of period \( T=5 \text{ s} \). We observe in these figures that the present method is able to accurately model the details of the wave field both in the water column, and in the porous-medium, where it exhibits fast attenuation in depth.

In the same case, the moduli of the modal-amplitude functions \( |P_n(x)| \) are plotted in Fig. 10 vs. the mode-number \( (n) \), as calculated by the present CMS. The horizontal axis in this figure is a multiple replica of the interval \( [a, b] \), i.e. a sequence of repeated intervals \( [a, b] \), each one associated with a mode, and named after the mode number. In the \( n \)th replica of interval \( [a, b] \) the amplitude of the \( n \)th mode is plotted. In order to illustrate the role and significance of the additional, sloping-interface mode \( P_{-1} \), results of two calculations are shown in Fig. 10. The one shown using thick lines is based on the present enhanced local mode series, Eq. (41), and the other one is obtained by means of the standard series, Eq. (28), not including the additional mode. In the same figure the curves bounding the maxima of \( |P_n(x)| \) are drawn by using thick and thin dashed lines, respectively. This result shows that the magnitude of the additional mode \( (n=-1) \) is of the same order as the first evanescent mode \( (n=1) \). Also, the rate of decay of the modal amplitudes of the enhanced series, Eq. (41), is \( O(n^{-4}) \), corresponding to very fast convergence. On the contrary, if the additional mode is not included, the decay is only \( O(n^{-2}) \) and thus, the convergence of the series (28) is poor. The above finding ensures the absolute and uniform convergence of the enhanced series (41), and its derivatives, up to \( (\text{and including}) \) the boundaries, and permits us to obtain accurate solutions, in the general bathymetry case, by keeping a small number of evanescent modes \( (e.g., 5-6 \text{ totally terms}) \) in the truncated local-mode series approximation, even for sea-bed slopes of the order \( 100\% \) and greater. We also observe in Fig. 10 the significance of the propagating mode, which is order(s) of magnitude greater than the rest of the modes. This fact permits us to obtain an improved mild-slope type model for the problem under consideration, based on the single-term approximation

\[
p(x, z) \approx P_0(x)2\sigma(\alpha z; x),
\]

based on the propagating mode. In this case the present CMS (48), neglecting all terms except \( n=0 \), reduces to following one-equation model:

\[
 a_{00}(x) \frac{d^2 P_0(x)}{dx^2} + b_{00}(x) \frac{dP_0(x)}{dx} + c_{00}(x) P_0(x) = 0, \quad a < x < b,
\]

for the determination of the unknown modal amplitude \( P_0(x) \), where the coefficients are defined through Eqs. (46) for \( n=m=0 \). The applicability of the above model will be demonstrated in the next example, illustrating also its usefulness.

Furthermore, groundwater flow velocities are obtained from pressure through Eq. (14), and in the examples of Figs. 7 and 8 they exhibit a fast decay in depth their maximum values being a small fraction of the corresponding wave velocities in the water column near the sea bed.

One important aspect of the present coupled model is its ability to provide information concerning the normal flow exchange rate between the water and the porous medium,
through the sea bottom boundary. This quantity is important for further studies, concerning, e.g., biological development of microorganisms in the upper layers of porous sediment of the coastal and onshore zone. Such a calculation, in the case of the smooth shoal environment (Eq. (55)), is presented in Fig. 11. We are able to observe in this figure that the modulus of the flow rate exchange between water and porous medium, as predicted by the present model, increases by 20–30% in the shallow region, which is attributed to shoaling effects. Such results could be found to be quite useful, in further studies examining the effects of waves on biodiversity of sandy beaches.

7.2. Case of a permeable rippled bed

Another example concerns propagation of water waves over permeable rippled bed topographies. Such cases are interesting in quantifying the effects of porosity on the Bragg scattering phenomena and have been considered by various authors, as e.g. Mase et al. (1995) and more recently by Huang et al. (2011).

One indicative example presented in Fig. 12 concerns waves of period $T = 1.3\, \text{s}$ and amplitude $A = 1.5\, \text{cm}$ propagating in water layer of average depth $h_{av} = 0.313\, \text{m}$, overlying porous layer of thickness 0.2m. In this case the wavy seabed consists of 10 simple sinusoidal ripples of wavelength $l_b = 1\, \text{m}$ ($k_b = 2\pi$) and amplitude $d = 0.16h_{av}$. The porosity is reported to be $n_w = 0.4$ and in this example the friction factor (as defined by Mase et al. (1995)) is taken $f = 0.1$. In order to show comparisons with results from our method, we use $E = E \approx 1.9 \times 10^9\, \text{Nm}^{-2}$ and estimate the corresponding value of coefficient of filtration (permeability) as $K_f \approx n_w^2 f/g = 0.19\, \text{ms}^{-1}$.

Our results presented in Fig. 12 are calculated by the present CMS, using $N_t = 3$ modes and $N = 300$ segments to subdivide the domain. Also, groundwater dynamic flow velocities, as obtained from pressure gradient through Eq. (14), are illustrated in the same plot using arrows. In this case, wave conditions are very close to Bragg resonance ($2k/l_b = 1$) and thus, the wave at the exit ($x = b = 10\, \text{m}$) is significantly attenuated due to multiple scattering effects.

This fact is also illustrated in Fig. 13 where the variation of the wave amplitude across the domain is plotted, as calculated by the present method (thick lines), in comparison with corresponding results obtained from the wave equation developed by Mase et al.
The latter model extends the theory of Kirby (1986) to waves propagating over permeable rippled beds. Two results obtained by the present system are shown in Fig. 13. The first one concerns the porous rippled bed and is shown by using thick solid lines. The second shown by thick dashed lines is obtained by the same system using zero porosity ($a + ib \to 0$), in which case our CMS reduces to the version corresponding to propagation of waves over impermeable bottom of general shape (see Athanassoulis and Belibassakis, 1999). In the latter case our results are found to be in noticeable agreement with the ones obtained by Mase et al. (1995) model for zero porosity, which is the same as Kirby's (1986) model. However, in the case of porous rippled bed our model predicts somewhat smaller attenuation (higher amplitude) of the wave as the end
reflected energy, for frequencies lower that resonance frequency, and this effect is reversed in the range of higher frequencies. This trend, at least in the case of impermeable bottom, seems to be confirmed by experimental data, as e.g. the ones presented by O’Hare and Davies (1993, Fig. 3c).

8. Conclusions

Beach groundwater-swash interaction is a subject of interest in coastal engineering, sediment transport and groundwater circulation. In this work, an enhanced coupled-mode system (CMS) is developed for modelling wave-induced pressure and groundwater flow in variable bathymetry coastal regions, in the layer under the permeable seabed. Focusing on the region where water depths are greater than the wave breaking depth, our method is based on a simplified version of the storage and momentum equations, appropriate in the case of very small elastic deformation of the porous medium. However, pore-water exchange flow effects are taken into account and the sea-bottom has been assumed to have general shape, possibly characterised by sloping parts and undulations. Results obtained by the CMS are found to be in reasonable agreement with other models and measured data. Thus, the present method, extending previous analysis concerning water-wave propagation in variable bathymetry domains to permeable bottom, could provide useful information concerning wave-induced groundwater dynamic pressure and flow in variable bathymetry coastal regions and sandy beaches. Future work is planned towards the detailed investigation and validation of the present model through its application to engineering and environmental studies and comparisons with established numerical models and experimental data, and finally, its extension to include elastic seabed deformation effects.

References


