Chapter 1
A coupled-mode theory for the diffraction of water waves by localized scatterers over a parallel-contour bathymetry

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Abstract

The consistent coupled-mode theory derived by Athanassoulis & Belibassakis [1] is generalized in (2+1) dimensions and applied to the diffraction problem of small-amplitude water waves from localized 3D scatterers superimposed over a parallel-contour bathymetry. The total wave field is decomposed into an incident field carrying out the effects of the parallel-contour bathymetry, and an associated diffraction field with forcing restricted only on the surface of the localized scatterer(s). The wave parameters in the horizontal space are different at infinity in different directions, rendering the estimation of the diffraction far-field pattern a part of the problem. In the representation of the vertical distribution of the diffraction potential an additional mode is introduced accounting for the bottom boundary condition when the bottom slope is not negligible. To restrict the support of the diffraction problem, an absorbing layer (the Berenger PML model) is introduced and optimized by direct minimization of the reflection coefficient. Numerical results are also presented.

1 Introduction
The interaction of free-surface gravity waves with an uneven bottom topography is a problem complicated by the fact that the physical properties of the medium (the wavenumber and propagation speed) are different at infinity in different directions and thus a standard radiation condition cannot be applied. In the present work, in order to describe the wave field when the bottom topography contains steep elements and/or localized scatterers, the consistent coupled-mode formulation [1], [2] is applied to the complete linear problem of propagation-refraction-diffraction. The examined environment consists of a transition region lying between two areas of constant but different depth, and containing the localized 3D scatterer(s); see Fig. 1. The problem is decomposed to an incident field carrying out the effects of the parallel-contour bathymetry, and a diffraction field carrying out the effects of the localized scatterer(s). The incident field is calculated separately, as a first step in the solution procedure, by applying the consistent coupled-mode theory [1] to the case of an obliquely incident monochromatic wave in the absence of the localized scatterer(s). In this way, the forcing associated with the diffraction problem is restricted only to the support of the localized scatterer(s). To tackle the problem of the unbounded domain, the perfectly matched layer (PML) technique, introduced by Berenger [3] for the absorption of electromagnetic waves, and formulated for general elliptic equations in [4],[5], is applied as a closure condition to the diffraction problem.

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2 Formulation of the problem

The studied marine environment consists of a water layer $D$ bounded above by the free surface $\partial D_F$ and below by a rigid bottom $\partial D_{II}$; see Fig. 1. It is assumed that the depth exhibits an arbitrary variation $h_2(x,y)$ in a bounded subdomain $D^{(2)}_C$, containing the support of the localized scatterer(s) superimposed over a parallel bottom contour. Outside this area, the bathymetry is characterized by an 1D variation $h_i(x)$. Thus, the depth function can be decomposed as $h_2(x,y) = h_i(x) + h_d(x,y)$, where $h_d(x,y)$ is the disturbance of the parallel bottom contour bathymetry $h_i(x)$ produced by the localized scatterer(s); see Fig. 1. Assuming that the free-surface elevation and the velocity field are small, the linearised wave equations can be used. The problem, formulated in terms of the frequency-dependent potential, is forced by the oblique-incident wave $\phi_oi(x,y,z)$ in the deep-water region with angle of incidence $\theta_i$. The total wave potential can be decomposed in two parts, the incident wave field $\phi_i(x,y,z)$ representing the propagation over the parallel-contour surface $h_i(x)$, and the diffraction field $\phi_d(x,y,z)$ generated by the 3D localized bathymetric features $h_d(x,y)$. Based on physical grounds, we assume that $\phi_i(x,y,z)$ is $y$-periodic with the same wavelength $\lambda = 2\pi/\kappa_y$, as the oblique-incident wave, where $\kappa_y = k_0^{(1)} \sin \theta_i$ and $k_0^{(1)}$ is the positive root of the dispersion relation $\mu h_1 = k_0^{(1)} h_1 \tanh(k_0^{(1)} h_1)$, $\mu = \omega^2 / g$. By introducing separation of variables $\phi_i(x,y,z) = \exp(j \kappa_y y) \varphi_i(x,z)$, $j = \sqrt{-1}$, we obtain the following 2D problem on $\varphi_i(x,z)$:

\[
\begin{align}
(1a) \quad \nabla^2 \varphi_i(x,z) - \kappa_y^2 \varphi_i(x,z) &= 0, \quad -\infty < x < \infty, \quad -h_i(x) < z < 0, \\
(1b) \quad \frac{\partial \varphi_i(x,z)}{\partial z} - \mu \varphi_i(x,z) &= 0, \quad -\infty < x < \infty, \quad z = 0, \\
(1c) \quad \left( \frac{\partial}{\partial x} + \nabla h \nabla \right) \varphi_i(x,z) &= 0, \quad -\infty < x < \infty, \quad z = -h_i(x),
\end{align}
\]

where $\nabla$ is the horizontal gradient operator. The problem is supplemented by proper radiation conditions [2],[6] and it is treated by means of the consistent coupled-mode theory, [1],[2]. By using the decomposition $\phi = \phi_i + \phi_d$ we obtain the following problem on the diffraction potential $\phi_d(x,y,z)$:

\[
\begin{align}
(2a) \quad \left( \frac{\partial^2}{\partial x^2} + \nabla^2 \right) \phi_d(x,y,z) &= 0, \quad (x,y,z) \in D, \\
(2b) \quad \frac{\partial \phi_d(x,y,z)}{\partial x} - \mu \phi_d(x,y,z) &= 0, \quad x, y, z \in \partial D_F, \\
(2c) \quad \left( \frac{\partial}{\partial x} + \nabla h \nabla \right) \phi_d(x,y,z) &= g(x,y) = -\left( \frac{\partial}{\partial x} + \nabla h \nabla \right) \phi_i, \quad (x,y,z) \in \partial D_{II},
\end{align}
\]

Fig. 1. Domain decomposition, coordinate system and basic notation.
supplemented by the following radiation conditions: In each of the two sectors \( D^{(m)}, m = 1,3 \), not including the bottom irregularity (see Fig. 1), the asymptotic behavior of the diffraction field approaches the form of an outgoing cylindrical wave, propagating with wavelength corresponding to the sector-depth at infinity \([2], i.e. \phi_d = O(R^{-1/2}), \partial \phi_d/\partial R - ik^{(m)}_0 \phi_d = O(R^{-3/2}), R \to \infty, r \in D^{(m)}, m = 1,3 \). Along the bottom irregularity, the diffraction field presents a faster decay \( \phi_d = O(R^{-3/2}), R \to \infty, r \in D^{(2)}, [2] \). On the basis of the definition of \( \phi_i \) potential, Eq. (1c), it is obvious that the support of the forcing \( g(x,y) \) of diffraction problem (2) is exactly the same as the support of the localized scatterer(s).

3 The coupled-mode system and the PML-model for the diffraction problem

The diffraction field is represented by the following enhanced local-mode series

\[
(3) \quad \phi_d(x, y, z) = \varphi_{-1}(x, y) Z_{-1}(z; x, y) + \varphi_0(x, y) Z_0(z; x, y) + \sum_{n=-1}^{\infty} \varphi_n(x, y) Z_n(z; x, y),
\]

including, except of the propagating mode and the evanescent modes, the additional term \( \varphi_{-1}(x) Z_{-1}(z; x) \), which is a correction term called the sloping-bottom mode, accounting for the bottom boundary condition on a sloping bottom, [1]. This representation is introduced in a variational principle equivalent to problem (2), leading to a system of coupled differential equations with respect to the modal functions \( \varphi_n(x, y), n = -1,0,1,2,... \), in the horizontal plane, [2]. In this way the 3D boundary value problem (2) is reformulated as a 2D problem, which is called the diffraction problem.

A difficulty associated with the solution of the diffraction problem is that the support of the modal amplitude functions \( \varphi_n(x, y) \) is the whole \( \mathbb{R}^2 \) and difficult radiation conditions should be applied. Since, however, far from the scatterers the propagating mode \( \varphi_0(x, y) \) will be at most of order \( O(R^{-1/2}) \), and the evanescent modes \( \varphi_n(x, y), n \neq 0 \), will exhibit an exponential decay, the wavelike behavior of the solution is substantially determined by the propagating \((n=0)\) mode. Accordingly, we can simplify the representation in the border area: \( x = \alpha_1, y = \alpha_2, y = \beta_1, y = \beta_2 \), by retaining there only the propagating mode \((n = 0)\). Also, in the vicinity of the border of the horizontal domain, the wavenumber \( k(x, y) \) can be considered constant. These simplifications permit us to couple the PML-model, [5], with the horizontal equations. We, thus, arrive at the following system of 2nd-order differential equations supported on the compact domain \([\alpha_1, \alpha_2] \times [\beta_1, \beta_2]\),

\[
(4) \quad \sum_{n=-1}^{\infty} a_{mn}(x, y) D^2 \varphi_n(x, y) + b_{mn}(x, y) \nabla \varphi_n(x, y) + c_{mn}(x, y) \varphi_n(x, y) = g_m(x, y),
\]

\( m = -1,0,1,... \), where the operator \( D^2 \) is defined by \( D^2 \varphi_n = \frac{1}{s_x} \left( \frac{\partial}{\partial x} \left( \frac{1}{s_x} \frac{\partial \varphi_n}{\partial x} \right) \right) + \frac{1}{s_y} \left( \frac{\partial}{\partial y} \left( \frac{1}{s_y} \frac{\partial \varphi_n}{\partial y} \right) \right) \), for \( n = 0 \) and \( D^2 \varphi_n = \nabla^2 \varphi_n \), for \( n \neq 0 \). Also, \( s_x \) is given by \( s_x = s(\tilde{x}) = 1 + \sigma(\tilde{x})/k, \sigma(\tilde{x}) > 0 \), where \( \tilde{x} = x - a_2 \), for \( x \in (a_2, a_2 + l) \), \( \tilde{x} = a_1 - x \), for \( x \in (a_1 - l, a_1) \), and \( l \) is the uniform thickness of the PML layer around the borderline of the horizontal domain \( D^{(2)}_C \). \( s_y \) is defined similarly. Both \( s_x \) and \( s_y \) are equal to unity outside the PML layer.

Using a polynomial form for the PML coefficient \( \sigma(\tilde{x}) = \sigma_0(\tilde{x}/l)^n, n \geq 3, [3] \), leads to a high-frequency approximation formula concerning the modulus of the reflection coefficient \( K_R \) of the form: \( |K_R| \propto \exp(-2 \cos \theta \cdot \sigma_0 l / (n + 1)) \), where \( \theta \) is the angle.
of incidence of the wave at the PML interface. From this we conclude that, for all angles of incidence except $\theta = \pi/2$, the reflection coefficient decays exponentially as the PML coefficient $\sigma (\tilde{x})$ increases. However, in discretizing the PML-equations by using a finite-difference scheme additional factors must enter into consideration, because of the undersmoothing produced by the discretization. It is shown in [4], that for a fixed number of points $N$ per wavelength ($\lambda = 2\pi/k$), the requirement of large $\sigma_0 > 0$ for good theoretical PML-efficiency must be balanced to the requirement of small rate of increase of $\sigma (\tilde{x})$, to ensure small numerical reflection. To optimize the discrete PML model we introduced the merit function

$$ R_m (n, \sigma_0 k; l/\lambda, N) = \int_{\theta=0}^{\theta=\pi/2} |R_{dB} (\theta; n, \sigma_0 k, l/\lambda, N)| (\cos \theta)^{1/4} d\theta, $$

where $R_{dB} = -20 \log_{10} |K_R|$ is the PML reflection coefficient calculated by using 2nd-order finite differences (based on $N$ grid points per wavelength). Then the corresponding optimization problems are solved with respect to $n$ and $\sigma_0 k$, when the parameters $l/\lambda$ and $N$ take specific values in the ranges $l/\lambda \in [0.5, 2]$ and $N \in [7, 20]$. The results of the optimization problem, valid for all values of the wavenumber $k$ are:

$$ n = 3, \quad \sigma_0 k = \gamma_1 (N) (l/\lambda)^3 \quad \text{and} \quad R_{m,opt} = \gamma_2 (N) \sqrt{l/\lambda} \quad \text{(in dB), for } l/\lambda \in [0.5, 2], $$

where the coefficients $\gamma_1, \gamma_2$ are dependent on $N$ and can be found in [2]. Using the present PML model with a typical thickness $l/\lambda = 1$ and grid resolution $N = 10$ the reflection coefficient remains below 1% (or -40dB) for angles of incidence up to almost $\theta = 80^\circ$.

4 Numerical results and discussion

As an example, we consider here the scattering by an elliptic shoal superimposed over a sloping bottom of constant slope 2°, [7]. For this environment, experimental data are available for an incident wave of period $T = 1sec$, propagating at an angle $\theta_1 = 20^\circ$. The calculated modulus of the total wave potential on the free-surface is presented in Fig. 2. In Fig. 3 the equipotential lines of the real part of the wave field in the variable bathymetry subdomain along the centerline of the elliptic shoal ($y = 0$) are presented, together with the calculated values of the wave potential on the free-surface which is proportional to free-surface elevation. Numerical results shown have been obtained by retaining 5 modes in the representation (the propagating mode, the sloping-bottom mode and the first 3 evanescent modes), which was found enough for numerical convergence. We can see in Fig. 3 that the equipotential lines intersect the bottom profile perpendicularly, as they ought. Comparisons between present model predictions and laboratory data concerning the normalized wave height are presented in Fig. 4 along various sections of Fig. 2. Finally, in Fig. 5 a comparison is presented concerning the calculated horizontal velocities along the centerline of the scatterer at the bottom, for an incident wave height $H = 0.1m$. The results that have been obtained by retaining only the propagating mode ($n = 1$) in the representations for the incident and the diffraction field (which is equivalent to the use of the modified mild-slope equation) are drawn by using dashed lines, and the results obtained by retaining 5 modes in the representation are drawn by using solid lines. As we can see, even in this case, where the maximum bottom slope does not exceeds locally 18%, the differences in the bottom velocities, as predicted by the present method, can be as high as 10%.
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Fig. 2. Bathymetry of the elliptic shoal over a sloping bottom from [7], and contours of the modulus of the total wave field $|\varphi|$ on the free surface.

Fig. 3. Equipotential lines of the real part of the wave field and free-surface elevation along the centerline of the scatterer.

Fig. 4. Normalized wave height along various sections of Fig. 2. Comparison between the experimental data [7] and numerical results.

Fig. 5. Moduli of horizontal bottom velocities along the centerline of the scatterer.

References


