On the numerical modelling of a shallow water equation

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Abstract - An implicit finite-difference method is presented for the numerical solution of the two-dimensional Boussinesq-type set of equations as those were introduced by Peregrine [PER,67]. The application of the finite-difference scheme results in an initial value problem and it is proposed the unknown quantities to be evaluated implicitly by solving a linear system of equations. The numerical treatment of the system is briefly discussed, while numerical results are the subject of a following work.

Keywords—**Shallow water waves; Boussinesq equations; Numerical modelling; Finite-Difference Method.**
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I. INTRODUCTION

During the last three decades a lot of effort has been put from the scientific community on numerical modelling of short waves in shallow water. Most of the phase-resolving models dealing with this research aspect and used in practical applications are based either on the mild-slope equation, originally derived by Berkhoff [BER,92], which describes the motion of time harmonic water waves of infinitesimal height (linear waves) on a gently sloping bathymetry with arbitrary water depth, or based on the Boussinesq-type equations governing the propagation of arbitrary, long-wave disturbances of small to moderate amplitude over a slowly-varying bathymetry, the first such set of which was derived by Peregrine [PER,67]. In case that the phenomenon of interest is the energy transfer among the different components of shoaling waves as the those propagate from offshore inshore into shallow water, then Boussinesq-type equations - which are weakly nonlinear and dispersive - have been found to model the nonlinear effects of the wave-transformation process with satisfactory accuracy. These equations are depth integrated, simplifying the full three-dimensional problem to a two-dimensional one, and can be formulated either in the time-domain (most usual) or in the frequency domain.

Most of the works dealing with numerical modelling in the time-domain employ Finite Difference Methods (FDM). Characteristic pioneer works on FDM concerning short-wave modelling in shallow water were that of Abbott *et al.* [ABT,78] according to which the differential equations were discretized by using a time-centered implicit scheme with variables defined on a space-staggered rectangular grid, as well as the one of Abbott *et al.* [ABT,84], including developments of the aforementioned scheme. Based on the same numerical method, Madsen *et al.* [MADS,91] presented a new form of equations in terms of the depth-integrated velocities for two horizontal dimensions with improved linear shoaling and dispersion characteristics and Madsen and Sørensen [MADS,92] rederived the new Boussinesq equations for slowly varying bathymetry. Beji and Battjes [BEJI,94] used similar equations and scheme to model relatively long, unidirectional waves propagating over a submerged obstacle. Wei and Kirby [WEI,95] developed a high-order numerical FDM scheme to solve a set of highly non-linear Boussinesq-type equations, while Beji and Nadaoka [BEJI,96] used three-time-level finite-difference approximations to model the corrected (energy conserving) equations of Madsen and Sorensen [MADS,92]. For general, extended reviews on the Boussinesq-type modelling the reader may have resource to the works of Madsen and Schäffer [MADS,99] and Kirby [KIR,03].

The present paper is based on the formulation of Madsen *et al.* [MADS,97], which, additionally to the work of Madsen and Sørensen [MADS,92], offers the possibility of wave breaking and a moving boundary at the shoreline. This work is not focused in the physical features, modelling or parameterization of the studied problem, since those are considered given and refer to the aforementioned work of Madsen *et al.* [MADS,97]. The subject is a finite-difference scheme and a proposed method for solving the two-dimensional Boussinesq-type equations, which result neglecting the terms expressing the effect of the bottom friction.

The organization of the paper is as follows. First, the physical problem is set and its governing equations are stated in Section II. Next, the numerical scheme is presented in Section III. The grid and the solution vectors are determined, and the finite-difference scheme is introduced and exploited. Last, in Section IV a predictor-corrector method for the solution of the obtained initial value problem is proposed and the future continuation of the work is shortly discussed.

II. GOVERNING EQUATIONS

Following Peregrine [PER,67] the equations of motion describing relatively long, small amplitudes waves propagating in water of varying depth are given by

$$\frac{\partial \zeta}{\partial t} + \boldsymbol{\nabla} \cdot \left[(h+\zeta)\, \mathbf{u} \right] = 0, \tag{1}$$

$$\mathbf{u}_{\,t} + (\mathbf{u} \cdot \boldsymbol{\nabla})\, \mathbf{u} + g \nabla \zeta$$

$$= \frac{1}{2}\, h \, \frac{\partial}{\partial t} \nabla \left[\boldsymbol{\nabla} \cdot (h\, \mathbf{u}) \right] - \frac{1}{6} h^2 \frac{\partial}{\partial t} \nabla \left(\boldsymbol{\nabla} \cdot \mathbf{u} \right) \tag{2}$$

in the region
$\Omega = \left\{ (x,y)\, ;\; L_x^0 < x < L_x^1,\; L_y^0 < y < L_y^1 \right\}$ for $t > 0$ where $\zeta = \zeta\,(x,y,t)$ is the surface displacement as it is measured from still water level and $\mathbf{u} = \mathbf{u}\,(x,y,t) = p\,(x,y,t)\,\mathbf{i} + q\,(x,y,t)\,\mathbf{j}$ is the depth-averaged horizontal velocity vector both sufficient differentiable functions, $\boldsymbol{\nabla} = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j}$, $h = h\,(x,y)$ is the still water depth and g the gravitational acceleration.

Using (1)-(2) and the assumptions introduced by Beji and Battjes [BEJI,94] for the one-dimensional problem $h_{xx} = h_{yy} = h_{xy} = 0$, system (1)-(2) finally reads to

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \left[(h+\zeta)\; p \right]}{\partial x} + \frac{\partial \left[(h+\zeta)\; q \right]}{\partial y} = 0, \tag{3}$$

$$\frac{\partial p}{\partial t} + p \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g\, \frac{\partial \zeta}{\partial x}$$

$$= \frac{1}{3} h^2 \, \frac{\partial^3 p}{\partial t\, \partial x^2} + h\, \frac{\partial h}{\partial x}\, \frac{\partial^2 p}{\partial t\, \partial x} + \frac{1}{2}\, h\, \frac{\partial h}{\partial x}\, \frac{\partial^2 q}{\partial t\, \partial y}$$

$$+ \frac{1}{2}\, h\, \frac{\partial h}{\partial y}\, \frac{\partial^2 q}{\partial t\, \partial x} + \frac{1}{3}\, h^2\, \frac{\partial^3 q}{\partial t\, \partial x\, \partial y}, \tag{4}$$

$$\frac{\partial q}{\partial t} + q \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g\, \frac{\partial \zeta}{\partial y}$$

$$= \frac{1}{3} h^2 \, \frac{\partial^3 q}{\partial t\, \partial y^2} + h\, \frac{\partial h}{\partial y}\, \frac{\partial^2 q}{\partial t\, \partial y} + \frac{1}{2}\, h\, \frac{\partial h}{\partial y}\, \frac{\partial^2 p}{\partial t\, \partial x}$$

$$+ \frac{1}{2}\, h\, \frac{\partial h}{\partial x}\, \frac{\partial^2 p}{\partial t\, \partial y} + \frac{1}{3}\, h^2\, \frac{\partial^3 p}{\partial t\, \partial x\, \partial y}\,. \tag{5}$$

Following an analogous scheme to the corresponding one dimensional problem to improve dispersion for the case of mildly sloping bottoms the terms

$$b h^2 \left(\frac{\partial^3 p}{\partial t\, \partial x^2} + g \frac{\partial^3 \zeta}{\partial x^3} \right) \tag{6}$$

and

$$b h^2 \left(\frac{\partial^3 q}{\partial t\, \partial y^2} + g \frac{\partial^3 \zeta}{\partial y^3} \right), \tag{7}$$

where b is the *calibration* factor, are added to the right-hand side of (4)-(5) respectively. Finally the system (3)-(5) is written as

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \left[(h+\zeta)\; p \right]}{\partial x} + \frac{\partial \left[(h+\zeta)\; q \right]}{\partial y} = 0, \tag{8}$$

$$\frac{\partial p}{\partial t} + p \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g\, \frac{\partial \zeta}{\partial x} = \tilde{b} h^2 \, \frac{\partial^3 p}{\partial t\, \partial x^2}$$

$$+ h\, \frac{\partial h}{\partial x}\, \frac{\partial^2 p}{\partial t\, \partial x} + \frac{1}{2}\, h\, \frac{\partial h}{\partial x}\, \frac{\partial^2 q}{\partial t\, \partial y}$$

$$+ \frac{1}{2}\, h\, \frac{\partial h}{\partial y}\, \frac{\partial^2 q}{\partial t\, \partial x} + \frac{1}{3}\, h^2\, \frac{\partial^3 q}{\partial t\, \partial x\, \partial y} + g \frac{\partial^3 \zeta}{\partial x^3}, \tag{9}$$

$$\frac{\partial q}{\partial t} + q \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g\, \frac{\partial \zeta}{\partial y} = \tilde{b} h^2 \, \frac{\partial^3 q}{\partial t\, \partial y^2}$$

$$+ h\, \frac{\partial h}{\partial y}\, \frac{\partial^2 q}{\partial t\, \partial y} + \frac{1}{2}\, h\, \frac{\partial h}{\partial y}\, \frac{\partial^2 p}{\partial t\, \partial x}$$

$$+ \frac{1}{2}\, h\, \frac{\partial h}{\partial x}\, \frac{\partial^2 p}{\partial t\, \partial y} + \frac{1}{3}\, h^2\, \frac{\partial^3 p}{\partial t\, \partial x\, \partial y} + g \frac{\partial^3 \zeta}{\partial y^3}\,, \tag{10}$$

where $\tilde{b} = b + \frac{1}{3}$ (see Madsen [MADS,91]-[MADS,92]).

III. THE PROPOSED NUMERICAL METHOD

A. Grid and solution vectors

To obtain a numerical solution the region $R = \Omega \times [t > 0]$ with its boundary ∂R consisting of the lines $x = L_x^0,\ L_x^1,\ y = L_y^0,\ L_y^1$ and $t = 0$, is covered with a rectangular mesh, G, of points with coordinates $(x, y, t) = (x_k, y_m, t_n) = \left(L_x^0 + kh_x, L_y^0 + mh_y, n\ell\right)$ with $k, m = 0, 1, ..., N+1$ and $n = 0, 1, ...,$ in which $h_x = \left(L_x^1 - L_x^0\right)/(N+1)$ and $h_y = \left(L_y^1 - L_y^0\right)/(N+1)$ represent the discretization into $N+1$ subintervals of the space variables, while ℓ represents the discretization of the time variable. The solution for the unknown functions ζ, p and q of an approximating finite-difference scheme at the same point will be denoted by $\zeta_{k,m}^n$, $p_{k,m}^n$ and $q_{k,m}^n$ respectively, while for the purpose of analyzing stability, the numerical value of actually obtained (subject, for instance, to computer round-off errors) will be denoted by $\tilde{\zeta}_{k,m}^n$, $\tilde{p}_{k,m}^n$ and $\tilde{q}_{k,m}^n$.

Let the solution vectors be

$$\boldsymbol{\zeta}^n = \boldsymbol{\zeta}(t_n) = \left[\zeta_{1,1}^n, \zeta_{1,2}^n, ..., \zeta_{1,N}^n;\ \zeta_{2,1}^n, \zeta_{2,2}^n, ..., \zeta_{2,N}^n;\right.$$

$$\left. ...;\ \zeta_{N,1}^n, \zeta_{N,2}^n, ..., \zeta_{N,N}^n\right]^T, \tag{11}$$

$$\mathbf{p}^n = \mathbf{p}(t_n) = \left[p_{1,1}^n, p_{1,2}^n, ..., p_{1,N}^n;\ p_{2,1}^n, p_{2,2}^n, ..., p_{2,N}^n;\right.$$

$$\left. ...;\ p_{N,1}^n, p_{N,2}^n, ..., p_{N,N}^n\right]^T, \tag{12}$$

$$\mathbf{q}^n = \mathbf{q}(t_n) = \left[q_{1,1}^n, q_{1,2}^n, ..., q_{1,N}^n;\ q_{2,1}^n, q_{2,2}^n, ..., q_{2,N}^n;\right.$$

$$\left. ...;\ q_{N,1}^n, q_{N,2}^n, ..., q_{N,N}^n\right]^T \tag{13}$$

T denoting transpose. Then there are N^2 values to be determined at each time step.

B. The initial conditions

The initial conditions are considered to be

$$\zeta(x, y, t_0) = \zeta_0(x, y), \tag{14}$$

$$p(x, y, t_0) = p_0(x, y), \tag{15}$$

$$q(x, y, t_0) = q_0(x, y), \tag{16}$$

when $x,\ y \in \Omega$. The values of $\boldsymbol{\zeta}_0$, $\mathbf{p}_0$ and $\mathbf{q}_0$ are in general unknown. For the numerical experiments they are assumed to be zero. It is known from the bibliography that due to frictional dissipation, the influence of the initial conditions becomes negligible after some time and stable conditions are appeared as the computation proceeds.

C. The boundary conditions

Following Bratsos [BRA,98]-[BRA,01] in (8)-(10) the boundary conditions will be assumed to be of the form

$$\frac{\partial f\left(L_x^0, y, t\right)}{\partial x} = \frac{\partial f\left(L_x^1, y, t\right)}{\partial x} = 0\,;,\ \ t > 0, \tag{17}$$

and

$$\frac{\partial f\left(x, L_y^0, t\right)}{\partial y} = \frac{\partial f\left(x, L_y^1, y, t\right)}{\partial y} = 0\,;\ \ t > 0, \tag{18}$$

where $f = \zeta,\ p,\ q$. Eqs. (17)-(18) give to first order

$$f\left(L_x^0 - h_x, y, t\right) = f\left(L_x^0 + h_x, y, t\right)\ \ ;\ \ t > t_0,$$

$$f\left(L_x^1 + h_x, y, t\right) = f\left(L_x^1 - h_x, y, t\right)\ \ ;\ \ t > t_0$$

or using the notation of the grid G,

$$f_{-1,\,m}^n = f_{1,\,m}^n, \tag{19}$$

$$f_{N+2,\,m}^n = f_{N,\,m}^n \tag{20}$$

for $m = 1, 2, ..., N$ and

$$f_{k,-1}^n = f_{k,1}^n, \tag{21}$$

$$f_{k,N+2}^n = f_{k,N}^n \tag{22}$$

for $k = 1, 2, ..., N$.

D. Development of the method

The following finite-difference formulae will be used

$$\frac{\partial f}{\partial x} = \frac{1}{2h_x}\left(f_{k+1,m} - f_{k-1,m}\right), \tag{23}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2h_y}\left(f_{k,m+1} - f_{k,m-1}\right),$$

$$\frac{\partial^2 f}{\partial t \partial x} = \frac{1}{2h_x l}\left(f_{k+1,m}^{n+1} - f_{k-1,m}^{n+1} - f_{k+1,m}^n + f_{k-1,m}^n\right),$$

$$\frac{\partial^2 f}{\partial t \partial y} = \frac{1}{2h_y l}\left(f_{k,m+1}^{n+1} - f_{k,m-1}^{n+1} - f_{k,m+1}^n + f_{k,m-1}^n\right),$$

$$\frac{\partial^3 f}{\partial t \partial x^2} = \frac{1}{2h_x^2 l}\left(f_{k+1,m}^{n+1} - f_{k+1,m}^{n-1} - 2f_{k,m}^{n+1}\right.$$

$$\left. + 2f_{k,m}^{n-1} + f_{k-1,m}^{n+1} - f_{k-1,m}^{n-1}\right),$$

$$\frac{\partial^3 f}{\partial t \partial y^2} = \frac{1}{2h_y^2 l}\left(f_{k,m+1}^{n+1} - f_{k,m+1}^{n-1} - 2f_{k,m}^{n+1}\right.$$

$$\left.+2f_{k,m}^{n-1} + f_{k,m-1}^{n+1} - f_{k,m-1}^{n-1}\right),$$

$$\frac{\partial^3 f}{\partial t \partial x \partial y} = \frac{1}{4h_x h_y l}\left[\left(f_{k+1,m+1}^{n+1} - f_{k+1,m-1}^{n+1}\right.\right.$$

$$\left.-f_{k-1,m+1}^{n+1} + f_{k-1,m-1}^{n+1}\right) - \left(f_{k+1,m+1}^{n} - f_{k+1,m-1}^{n}\right.$$

$$\left.\left.-f_{k-1,m+1}^{n} + f_{k-1,m-1}^{n}\right)\right], \qquad (24)$$

where the latter one is second order in x, y and first order in t.

Then using the formulae (23)-(24) system (8)-(10), when applied to each interior point (x_k, y_m, t_n) of the grid G after using the Crank-Nickolson method, gives the following three-time level finite-difference scheme

$$\frac{1}{\ell}\left(\zeta_{k,m}^{n+1} - \zeta_{k,m}^{n}\right) + \frac{1}{2h_x}\left[\left(h_{k,m} + \zeta_{k,m}\right) p_{k+1,m}^{n}\right.$$

$$\left.-\left(h_{k,m} + \zeta_{k,m}\right) p_{k-1,m}^{n}\right] + \frac{1}{2h_y}\left[\left(h_{k,m} + \zeta_{k,m}\right) q_{k,m+1}^{n}\right.$$

$$\left.-\left(h_{k,m} + \zeta_{k,m}\right) q_{k,m-1}^{n}\right] = 0, \qquad (25)$$

$$-\frac{h_{k,m}^2}{12h_x h_y \ell}\left(q_{k+1,m+1}^{n+1} - q_{k+1,m-1}^{n+1}\right)$$

$$-\frac{h_{k,m}}{8h_x h_y \ell}\left(h_{k,m+1} - h_{k,m-1}\right) q_{k+1,m}^{n+1}$$

$$+\frac{1}{4h_y}\left[p_{k,m}^{n+\frac{1}{2}} - \frac{h_{k,m}}{2h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] q_{k,m+1}^{n+1}$$

$$+\frac{1}{4h_y}\left[-p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] q_{k,m-1}^{n+1}$$

$$+\frac{h_{k,m}^2}{12h_x h_y \ell}\left(q_{k-1,m+1}^{n+1} - q_{k-1,m-1}^{n+1}\right)$$

$$+\frac{h_{k,m}}{8h_x h_y \ell}(h_{k,m+1} - h_{k,m-1}) q_{k-1,m}^{n+1}$$

$$+\frac{1}{2h_x}\left[\frac{p_{k,m}^{n+\frac{1}{2}}}{2} - \tilde{b}\frac{h_{k,m}^2}{h_x \ell} - \frac{h_{k,m}}{2h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] p_{k+1,m}^{n+1}$$

$$+\left(\frac{1}{\ell} + \tilde{b}\frac{h_{k,m}^2}{h_x^2 \ell}\right) p_{k,m}^{n+1} + \frac{1}{2h_x}\left[-\frac{p_{k,m}^{n+\frac{1}{2}}}{2}\right.$$

$$\left.-\tilde{b}\frac{h_{k,m}^2}{h_x \ell} + \frac{h_{k,m}}{2h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] p_{k-1,m}^{n+1}$$

$$= -\frac{h_{k,m}^2}{12h_x h_y \ell}\left(q_{k+1,m+1}^{n} - q_{k+1,m-1}^{n}\right)$$

$$-\frac{h_{k,m}}{8h_x h_y \ell}\left(h_{k,m+1} - h_{k,m-1}\right) q_{k+1,m}^{n}$$

$$+\frac{h_{k,m}^2}{12h_x h_y \ell}\left(q_{k-1,m+1}^{n} - q_{k-1,m-1}^{n}\right)$$

$$-\frac{1}{4h_y}\left[p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] q_{k,m+1}^{n}$$

$$+\frac{1}{4h_y}\left[p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] q_{k,m-1}^{n}$$

$$+\frac{h_{k,m}}{8h_x h_y \ell}(h_{k,m+1} - h_{k,m-1}) q_{k-1,m}^{n}$$

$$-\frac{1}{4h_x}\left[p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{h_x \ell}\left(h_{k+1,m} - h_{k-1,m}\right)\right] p_{k+1,m}^{n}$$

$$+\frac{1}{4h_x}\left[p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{h_x \ell}(h_{k+1,m} - h_{k-1,m})\right] p_{k-1,m}^{n}$$

$$+\frac{1}{\ell} p_{k,m}^{n} - \tilde{b}\,\frac{h_{k,m}^2}{2h_x^2 \ell}\left(p_{k+1,m}^{n-1} - 2p_{k,m}^{n-1} + p_{k-1,m}^{n-1}\right)$$

$$-\frac{g}{4h_x}\left(\zeta_{k+1,m}^{n+1} - \zeta_{k-1,m}^{n+1} + \zeta_{k+1,m}^{n} - \zeta_{k-1,m}^{n}\right)$$

$$+\frac{g}{4h_x^3}\left(-\zeta_{k-2,m}^{n+1} + 2\zeta_{k-1,m}^{n+1} - 2\zeta_{k+1,m}^{n+1} + \zeta_{k+2,m}^{n+1}\right.$$

$$\left.+ -\zeta_{k-2,m}^{n} + 2\zeta_{k-1,m}^{n} - 2\zeta_{k+1,m}^{n} + \zeta_{k+2,m}^{n}\right), \qquad (26)$$

$$-\frac{h_{k,m}^2}{12h_x h_y \ell}\left(p_{k+1,m+1}^{n+1} - p_{k+1,m-1}^{n+1}\right)$$

$$+\frac{1}{4h_x}\left[q_{k,m}^{n+\frac{1}{2}} - \frac{h_{k,m}}{2h_y \ell}\left(h_{k,m+1} - h_{k,m-1}\right)\right] p_{k+1,m}^{n+1}$$

$$-\frac{h_{k,m}}{8h_x h_y \ell}\left(h_{k+1,m}-h_{k-1,m}\right)p^{n+1}_{k,m+1}$$

$$+\frac{h_{k,m}}{8h_x h_y \ell}\left(h_{k+1,m}-h_{k-1,m}\right)p^{n+1}_{k,m-1}$$

$$+\frac{h^2_{k,m}}{12h_x h_y \ell}\left(p^{n+1}_{k-1,m+1}-p^{n+1}_{k-1,m-1}\right)$$

$$+\frac{1}{4h_x}\left[-q^{n+\frac{1}{2}}_{k,m}+\frac{h_{k,m}}{2h_y \ell}(h_{k,m+1}-h_{k,m-1})\right]p^{n+1}_{k-1,m}$$

$$+\frac{1}{2h_y}\left[\frac{q^{n+\frac{1}{2}}_{k,m}}{2}-\tilde{b}\,\frac{h^2_{k,m}}{h_y \ell}\right.$$

$$\left.-\frac{h_{k,m}}{2h_y \ell}(h_{k,m+1}-h_{k,m-1})\right]q^{n+1}_{k,m+1}$$

$$+\left(\frac{1}{\ell}+\tilde{b}\frac{h^2_{k,m}}{h^2_y \ell}\right)q^{n+1}_{k,m}$$

$$+\frac{1}{2h_y}\left[-\frac{q^{n+\frac{1}{2}}_{k,m}}{2}-\tilde{b}\,\frac{h^2_{k,m}}{h_y \ell}\right.$$

$$\left.+\frac{h_{k,m}}{2h_y \ell}(h_{k,m+1}-h_{k,m-1})\right]q^{n+1}_{k,m-1}$$

$$=-\frac{h^2_{k,m}}{12h_x h_y \ell}\left(p^n_{k+1,m+1}-p^n_{k+1,m-1}\right)$$

$$-\frac{1}{4h_x}\left[q^{n+\frac{1}{2}}_{k,m}+\frac{h_{k,m}}{2h_y \ell}\left(h_{k,m+1}-h_{k,m-1}\right)\right]p^n_{k+1,m}$$

$$-\frac{h_{k,m}}{8h_x h_y \ell}\left(h_{k+1,m}-h_{k-1,m}\right)\left(p^n_{k,m+1}-p^n_{k,m-1}\right)$$

$$+\frac{h^2_{k,m}}{12h_x h_y \ell}\left(p^n_{k-1,m+1}-p^n_{k-1,m-1}\right)$$

$$+\frac{1}{4h_x}\left[q^{n+\frac{1}{2}}_{k,m}+\frac{h_{k,m}}{2h_y \ell}\left(h_{k,m+1}-h_{k,m-1}\right)\right]p^n_{k-1,m}$$

$$-\frac{1}{4h_y}\left[q^{n+\frac{1}{2}}_{k,m}+\frac{h_{k,m}}{h_y \ell}\left(h_{k,m+1}-h_{k,m-1}\right)\right]q^n_{k,m+1}$$

$$+\frac{1}{4h_y}\left[q^{n+\frac{1}{2}}_{k,m}+\frac{h_{k,m}}{h_y \ell}(h_{k,m+1}-h_{k,m-1})\right]q^n_{k,m-1}$$

$$+\frac{1}{\ell}q^n_{k,m}-\tilde{b}\,\frac{h^2_{k,m}}{2h^2_y \ell}\left(q^{n-1}_{k,m+1}-2q^{n-1}_{k,m}+q^{n-1}_{k,m-1}\right)$$

$$-\frac{g}{4h_y}\left(\zeta^{n+1}_{k,m+1}-\zeta^{n+1}_{k,m-1}+\zeta^n_{k,m+1}-\zeta^n_{k,m-1}\right)$$

$$+\frac{g}{4h^3_y}\left(-\zeta^{n+1}_{k,m-2}+2\zeta^{n+1}_{k,m-1}-2\zeta^{n+1}_{k,m+1}+\zeta^{n+1}_{k,m+2}\right.$$

$$\left.-\zeta^n_{k,m-2}+2\zeta^n_{k,m-1}-2\zeta^n_{k,m+1}+\zeta^n_{k,m+2}\right) \tag{27}$$

for $k, m = 1, 2, ..., N$ and $n = 1, 2, ...$

The term $f^{n+\frac{1}{2}}_m$ in (26)-(27) is determined by using Taylor's series expansion about (x, t) as follows

$$f\left(x, t+\frac{1}{2}\Delta t\right)=f\left(x,t\right)+\frac{1}{2}\,\Delta t\,\frac{\partial f\left(x,t\right)}{\partial t}$$

$$+O\left(\Delta t^2\right)\ \text{ as }\ \Delta t\to 0. \tag{28}$$

IV. THE PROPOSED SOLUTION AND SHORT DISCUSSION

Using the boundary conditions specified at the section III paragraph C the unknown vector $\boldsymbol{\zeta}^{n+1}$ can be evaluated explicitly from Eq. (25)

$$\boldsymbol{\zeta}^{n+1}=\boldsymbol{\zeta}^n+\ell\,\mathbf{F}_1\left(\mathbf{f}^{\,n},\mathbf{f}^{\,n-1}\right) \tag{29}$$

for $n = 0, 1, ...$ Eqs. (26)-(27) are written in a matrix-vector form as

$$A_1\,\mathbf{p}^{n+1}+A_2\,\mathbf{q}^{n+1}=\mathbf{F}_2\left(\mathbf{f}^{\,n},\mathbf{f}^{\,n-1}\right), \tag{30}$$

$$B_1\,\mathbf{p}^{n+1}+B_2\,\mathbf{q}^{n+1}=\mathbf{F}_3\left(\mathbf{f}^{\,n},\mathbf{f}^{\,n-1}\right), \tag{31}$$

where A_i, B_i; $i = 1, 2$ are matrices of order N^2 with appropriate entries, so the unknown vectors $\mathbf{p}^{n+1}$ and $\mathbf{q}^{n+1}$ are evaluated implicitly from the solution of the system (29)-(30) either by direct methods using familiar algorithms based on the *LU* decomposition or by Gauss-Seidel method.

Then the value $\boldsymbol{\zeta}^{n+1}$ evaluated from (29) is *corrected* using the values of $\mathbf{p}^{n+1}$, $\mathbf{q}^{n+1}$ obtained by (30)-(31) using the Crank-Nickolson method as follows

$$\zeta^{n+1}_m=\zeta^n_m-\frac{\ell}{4h_x}\,\Big\{\left[\left(h+\zeta\right)p\right]^{n+1}_{k+1,m}-\left[\left(h+\zeta\right)p\right]^{n+1}_{k-1,m}$$

$$+\left[\left(h+\zeta\right)p\right]^n_{k+1,m}-\left[\left(h+\zeta\right)p\right]^n_{k-1,m}\Big\}$$

$$-\frac{\ell}{4h_y}\left\{[(h+\zeta)\,p]^{n+1}_{k,m+1}-[(h+\zeta)\,p]^{n+1}_{k,m-1}\right.$$

$$\left.+[(h+\zeta)\,p]^{n}_{k,m+1}-[(h+\zeta)\,p]^{n}_{k,m-1}\right\} \qquad (32)$$

for $m=1,2,...,N$.

The above procedure is described from the following predictor-corrector algorithm

i)	**P :**	predict the value $\boldsymbol{\zeta}^{n+1}$ from (29),
ii)	**E :**	evaluate the values $\mathbf{p}^{n+1}$, $\mathbf{q}^{n+1}$ from (30)-(31),
iii)	**C :**	correct $\boldsymbol{\zeta}^{n+1}$ using (32),
iv)	**E :**	repeat the above procedure (i)-(iii) to achieve the desired accuracy.

The latter one and the numerical results of the method is a work under preparation, which is going to be sent for publishing soon.

A. Stability analysis

For the the proposed finite-difference scheme (25)-(27), the following condition is necessary for stability

$$\mathrm{Cf}\ <\ 1, \qquad (33)$$

where Cf is the Courant number. Condition (33) is the well-known CFL condition developed by Courant-Freiedrichs-Lewy [CFL,1928]. If the CFL condition is not satisfied, convergence can not be guaranteed (see also Mitchell and Griffiths [MTC,80] and Twizell [1]). For the two-dimensional shallow water equations the Courant number is defined as follows (see Stelling [STL,84])

$$\mathrm{Cf}=\ell\,\sqrt{g\,h_0}\,\left(\frac{1}{h_x^2}+\frac{1}{h_y^2}\right)^{1/2}, \qquad (34)$$

where

$$h_0=\max_{m=0,1,\ldots,N+1} h_m. \qquad (35)$$

Condition (34) using linearization (35) leads to the following criterion for the optimal time step (see also Leendertse [LND,67])

$$\ell\ \le\ \frac{h_x\,h_y}{\sqrt{g\,h_0\left(h_x^2+h_y^2\right)}}\,. \qquad (36)$$

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