An implicit numerical method for a shallow water equation in 2 + 1 dimensions

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An implicit finite-difference method is presented for the numerical solution of the two-dimensional Boussinesq-type set of equations as those were introduced by Peregrine [14]. The application of the finite-difference scheme results in an initial value problem and it is proposed the unknown quantities to be evaluated implicitly by solving a linear system of equations.

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1 Introduction

Following Peregrine [14] the equations of motion describing relatively long, small amplitudes waves propagating in water of varying depth are given by

\[
\frac{\partial \zeta}{\partial t} + \nabla \cdot [(h + \zeta) \mathbf{u}] = 0,
\]

\[
\mathbf{u}_t + (\mathbf{u} \cdot \nabla) \mathbf{u} + g \nabla \zeta = \frac{1}{2} h \frac{\partial}{\partial t} \nabla \cdot (h \mathbf{u}) - \frac{1}{6} h^2 \frac{\partial^2}{\partial t^2} \nabla \cdot (h \mathbf{u})
\]

in the region \( \Omega = \{(x, y); L_0 x < x < L_1 x, L_0 y < y < L_1 y\} \) for \( t > 0 \), where \( \zeta = \zeta(x, y, t) \) is the surface displacement as it is measured from the still water level and \( \mathbf{u} = \mathbf{u}(x, y, t) = p(x, y, t) \mathbf{i} + q(x, y, t) \mathbf{j} \) is the depth-averaged horizontal velocity vector both sufficient differentiable functions, \( \nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} \), \( h = h(x, y) \) is the still water depth and \( g \) the gravitational acceleration.

Using (1)-(2) and the assumptions introduced by Beji and Battjes [2] for the one-dimensional problem \( h_{xx} = h_{yy} = h_{xy} = 0 \), system (1)-(2) finally reads to

\[
\frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} \left[ (h + \zeta) p \right] + \frac{\partial}{\partial y} \left[ (h + \zeta) q \right] = 0,
\]

\[
\frac{\partial p}{\partial t} + p \left( \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g \frac{\partial \zeta}{\partial x} = \frac{1}{3} h^2 \frac{\partial^3 p}{\partial t \partial x^2} + h \frac{\partial h}{\partial x} \frac{\partial^2 p}{\partial t \partial x} + \frac{1}{2} h \frac{\partial h}{\partial y} \frac{\partial^2 q}{\partial t \partial x} + \frac{1}{2} h \frac{\partial h}{\partial y} \frac{\partial^2 q}{\partial t \partial y} + h^2 \frac{\partial^2 q}{\partial t \partial x \partial y},
\]

\[
\frac{\partial q}{\partial t} + q \left( \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g \frac{\partial \zeta}{\partial y}
\]

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\[ f(t, x, y) = \sum_{n=0}^{N} u_n (x, y, t) \]

Following an analogous scheme to the corresponding one dimensional problem to improve dispersion for the case of mildly sloping bottoms the terms \( bh^2 \left( \frac{\partial^3 q}{\partial x^3} + g \frac{\partial^3 \zeta}{\partial y^3} \right) \) and \( bh^2 \left( \frac{\partial^3 q}{\partial y^3} + g \frac{\partial^3 \zeta}{\partial x^3} \right) \), where \( b \) is the calibration factor, are added to the right-hand side of (4)-(5) respectively. Finally the system (3)-(5) is written as

\[ \begin{align*}
\frac{\partial \zeta}{\partial t} + \frac{\partial [(h + \zeta) p]}{\partial x} + \frac{\partial [(h + \zeta) q]}{\partial y} &= 0, \\
\frac{\partial p}{\partial t} + p \left( \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g \frac{\partial \zeta}{\partial y} &= \tilde{b} h^2 \frac{\partial^3 p}{\partial t \partial x^3}, \\
+ h \frac{\partial h}{\partial x} \frac{\partial^2 p}{\partial t \partial x} + \frac{1}{2} h \frac{\partial h}{\partial y} \frac{\partial^2 q}{\partial t \partial y} + \frac{1}{2} h \frac{\partial h}{\partial y} \frac{\partial^2 q}{\partial t \partial x} + \frac{1}{3} h^2 \frac{\partial^3 q}{\partial t \partial x \partial y} + g \frac{\partial^3 \zeta}{\partial x^3}, \\
\frac{\partial q}{\partial t} + q \left( \frac{\partial p}{\partial x} + \frac{\partial q}{\partial y} \right) + g \frac{\partial \zeta}{\partial x} &= \tilde{b} h^2 \frac{\partial^3 q}{\partial t \partial y^3}, \\
+ h \frac{\partial h}{\partial y} \frac{\partial^2 q}{\partial t \partial y} + \frac{1}{2} h \frac{\partial h}{\partial x} \frac{\partial^2 p}{\partial t \partial x} + \frac{1}{2} h \frac{\partial h}{\partial x} \frac{\partial^2 p}{\partial t \partial y} + \frac{1}{3} h^2 \frac{\partial^3 p}{\partial t \partial x \partial y} + g \frac{\partial^3 \zeta}{\partial y^3},
\end{align*} \]

where \( \tilde{b} = b + \frac{1}{3} \) (see Madsen [9]-[10]).

2 The numerical method

2.1 Grid and solution vectors

To obtain a numerical solution the region \( R = \Omega \times [t > 0] \) with its boundary \( \partial R \) consisting of the lines \( x = L_x^0, L_x^1, y = L_y^0, L_y^1 \) and \( t = 0 \), is covered with a rectangular mesh, \( G \), of points with coordinates \( (x, y, t) = (x_k, y_m, t_n) = (L_x^0 + kh_x, L_y^0 + mh_y, n\ell) \) with \( k, m = 0, 1, ..., N + 1 \) and \( n = 0, 1, ..., \) in which \( h_x = (L_x^1 - L_x^0) / (N + 1) \) and \( h_y = (L_y^1 - L_y^0) / (N + 1) \) represent the discretization into \( N + 1 \) subintervals of the space variables, while \( \ell \) represents the discretization of the time variable. The solution for the unknown functions \( \zeta, p \) and \( q \) of an approximating finite-difference scheme at the same point will be denoted by \( \zeta^n_{k,m}, p^n_{k,m} \) and \( q^n_{k,m} \) respectively, while for the purpose of analyzing stability, the numerical value of actually obtained (subject, for instance, to computer round-off errors) will be denoted by \( \hat{\zeta}_{k,m}, \hat{p}_{k,m} \) and \( \hat{q}_{k,m} \).

Let the solution vector for the function \( \zeta \) be

\[ \zeta^n = \zeta (t_n) = [\zeta^n_{1,1}, \zeta^n_{1,2}, ..., \zeta^n_{1,N}; \zeta^n_{2,1}, \zeta^n_{2,2}, ..., \zeta^n_{2,N}; \cdots; \zeta^n_{N,1}, \zeta^n_{N,2}, ..., \zeta^n_{N,N}]^T \]

\( T \) denoting transpose with analogous expressions for \( p \) and \( q \) respectively.

The initial conditions are considered to be

\[ \zeta (x, y, 0) = \zeta_0 (x, y), \quad p (x, y, 0) = p_0 (x, y) \quad \text{and} \quad q (x, y, 0) = q_0 (x, y), \]

when \( x, y \in \Omega \). The values of \( \zeta_0, \ p_0 \) and \( q_0 \) are in general unknown. It is known from the bibliography that due to frictional dissipation, the influence of the initial conditions becomes negligible after some time and stable conditions are appeared as the computation proceeds.

Following Bratsos [4]-[5] in (6)-(8) the boundary conditions will be assumed to be of the form

\[ \frac{\partial f (x, y, t)}{\partial x} = g (y, t) \quad \text{for} \quad x = L_x^0, L_x^1 \quad \text{and} \quad \frac{\partial f (x, y, t)}{\partial y} = \hat{g} (x, t) \quad \text{for} \quad y = L_y^0, L_y^1; \quad t > 0 \]

for \( f = \zeta, p, q. \)
2.2 Development of the method

Using familiar finite-difference replacements system (6)-(8), when applied to each interior point \((x_k, y_m, t_n)\) of the grid \(G\) after using the Crank-Nicolson method, gives the following three-time level finite-difference scheme

\[
\begin{align*}
\frac{1}{\ell} \left( \zeta_{k,m}^{n+1} - \zeta_{k,m}^n \right) + \frac{1}{2h_x} \left[ \left(h_{k,m} + \zeta_{k,m}\right) p_{k+1,m}^n - \left(h_{k,m} + \zeta_{k,m}\right) p_{k-1,m}^n \right] \\
+ \frac{1}{2h_y} \left[ \left(h_{k,m} + \zeta_{k,m}\right) q_{k,m+1}^n - \left(h_{k,m} + \zeta_{k,m}\right) q_{k,m-1}^n \right] = 0,
\end{align*}
\]  

(12)

\[
\begin{align*}
&- \frac{h_{k,m}^2}{12h_x h_y \ell} \left( q_{k+1,m+1}^{n+1} - q_{k+1,m-1}^{n+1} \right) - \frac{h_{k,m}}{8h_x h_y \ell} \left(h_{k+1,m} - h_{k,m-1}\right) q_{k+1,m}^{n+1} \\
&+ \frac{1}{4h_y} \left[ p_{k,m}^{n+\frac{1}{2}} - \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] q_{k,m+1}^{n+1} + \frac{1}{4h_y} \left[ -p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] q_{k,m-1}^{n+1} \\
&+ \frac{1}{2h_x} \left[ p_{k,m}^{n+\frac{1}{2}} - \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] \left[ n+1 \right] p_{k+1,m}^{n+1} + \left( \frac{1}{\ell} + \frac{b \ell^2}{h_x^2} \right) p_{k,m}^{n+1} \\
&+ \frac{1}{2h_x} \left[ -p_{k,m}^{n+\frac{1}{2}} - \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] \left[ n+1 \right] p_{k-1,m}^{n+1} \\
&= - \frac{h_{k,m}^2}{12h_x h_y \ell} \left( q_{k+1,m+1}^n - q_{k+1,m-1}^n \right) - \frac{h_{k,m}}{8h_x h_y \ell} \left(h_{k+1,m} - h_{k,m-1}\right) q_{k+1,m}^n \\
&+ \frac{h_{k,m}^2}{12h_x h_y \ell} \left( q_{k-1,m+1}^n - q_{k-1,m-1}^n \right) - \frac{1}{4h_y} \left[ p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] q_{k,m+1}^n \\
&+ \frac{1}{4h_y} \left[ p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] q_{k,m-1}^n + \frac{1}{4h_y} \left[ -p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] \left[ n+1 \right] p_{k+1,m}^n \\
&+ \frac{1}{4h_y} \left[ -p_{k,m}^{n+\frac{1}{2}} + \frac{h_{k,m}}{2h_x \ell} (h_{k+1,m} - h_{k,m-1}) \right] \left[ n+1 \right] p_{k-1,m}^n \\
&+ \frac{1}{\ell^2} \left[ p_{k,m}^n - \frac{b \ell^2}{2h_x^2} \right] \left( p_{k+1,m}^{n-1} - 2p_{k,m}^{n-1} + p_{k-1,m}^{n-1} \right) - \frac{g}{4h_x} \left( \zeta_{k-2,m}^{n+1} - \zeta_{k-1,m}^{n+1} + \zeta_{k+1,m}^{n+1} - \zeta_{k-1,m}^{n+1} \right) \\
&+ \frac{g}{4h_x^2} \left( -2\zeta_{k-2,m}^{n+1} + 2\zeta_{k-1,m}^{n+1} - 2\zeta_{k+1,m}^{n+1} + \zeta_{k+2,m}^{n+1} - \zeta_{k-2,m}^{n+1} + 2\zeta_{k-1,m}^{n+1} - 2\zeta_{k+1,m}^{n+1} + \zeta_{k+2,m}^{n+1} \right)
\end{align*}
\]  

(13)

for \(k, m = 1, 2, ..., N\) and \(n = 1, 2, ...\) and an analogous expression for Eq. (8). In Eq. (13) the term \(p_{k,m}^{n+\frac{1}{2}}\) is going to be determined by using Taylor’s series expansion about \((x_k, y_m, t_n)\).

Finally, using the boundary conditions defined in Eq. (11) for the evaluation of the unknown solution vectors the following linear system arises

\[
\zeta_{k,m}^{n+1} = \zeta_{k,m}^n + \ell \mathbf{F}_1 \left( \mathbf{f}^n, \mathbf{f}^{n-1} \right),
\]  

(14)

\[
A_1 \mathbf{p}^{n+1} + A_2 \mathbf{q}^{n+1} = \mathbf{F}_2 \left( \mathbf{f}^n, \mathbf{f}^{n-1} \right),
\]  

(15)
for \( n = 2, 3, \ldots \), where \( A_i, B_i; i = 1, 2 \) are matrices of order \( N^2 \) with appropriate entries.

Then the value \( \zeta^{n+1} \) evaluated explicitly from (14) is corrected using the values \( \mathbf{p}^{n+1}, \mathbf{q}^{n+1} \) obtained by (15)-(16) using the Crank-Nickolson method as follows

\[
\zeta_m^{n+1} = \zeta_m^n - \frac{\ell}{4h_x} \left\{ \left[ (h + \zeta) p_{k+1,m}^{n+1} - [(h + \zeta) p_{k-1,m}^{n+1} + [(h + \zeta) p_{k-1,m}^{n+1} \right] \\
- \frac{\ell}{4h_y} \left\{ \left[ (h + \zeta) p_{k,m+1}^{n+1} - [(h + \zeta) p_{k,m-1}^{n+1} + [(h + \zeta) p_{k,m-1}^{n+1} \right] \right\}
\]

for \( m = 1, 2, \ldots, N \).

The above procedure is described from the following predictor-corrector algorithm

i) P : predict the value \( \zeta^{n+1} \) from (14),

ii) E : evaluate the values \( \mathbf{p}^{n+1}, \mathbf{q}^{n+1} \) from (15)-(16)

iii) C : correct \( \zeta^{n+1} \) using (17),

iv) E : repeat the above procedure (i)-(iii) to achieve the desired accuracy.

The method was tested numerically and the results arised are discussed.

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