A coupled-mode technique for the prediction of wave-induced set-up in variable bathymetry domains and groundwater circulation in permeable beaches

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ABSTRACT

In the present work a complete phase-resolving wave model is coupled with an iterative solver of the mean-flow equations, permitting an accurate calculation of wave-induced set-up in intermediate and shallow water environments with possibly steep bathymetric variations. The wave model is based on the consistent coupled-mode system for the propagation of water waves in variable bathymetry regions, developed by Athanassoulis & Belibassakis (1999) and extended to 3D by Belibassakis et al (2001). This model improves the predictions of the mild-slope equation(s), permitting accurate treatment of wave propagation in regions with steep bottom slope and/or large curvature. In addition, it supports the correct calculation of wave velocity up to the bottom boundary. The coupled-mode model has been further extended to include the effects of energy dissipation from bottom friction and wave breaking, which are important for the accurate calculation of radiation stresses on decreasing depth. Furthermore, it has been used in conjunction with the mean flow equations to predict wave-induced set-up and flow in closed and open domains. Finally, the resulting phase-averaged mean-pressure has been applied to calculate the induced groundwater circulation on a permeable beach, in the set-up region. Under the assumption that the groundwater flow is in the Darcy law regime, in the case of a stationary mean flow, the porous flow velocity can be obtained in terms of the pressure gradient. In this case, Massel (2001), the problem concerning groundwater circulation is governed by the Laplace’s equation on the pressure gradient, forced by Dirichlet data specified by the excess pressure on the sea bottom that is induced by the mean flow.

KEY WORDS: coupled-modes, mean flow equations, porous flow

INTRODUCTION

Wave transformation on beaches along with other important factors, such as wave breaking, set-up and run-up, play a significant role in the water table formation and groundwater flow, and contributes to the dynamics of the coastal zone. Useful information concerning all the above effects can be obtained from the solution of the slow-scale, mean-flow equations, forced by radiation stresses as well as the freesurface and the bottom stresses. The radiation stresses can be effectively calculated from the fast-scale wave flow properties; see, e.g., Mei (1983), Dingemans (1997).

As concerns the interaction of free-surface gravity waves with variable bottom topography, in water of intermediate depth and in shallow water, a broad class of approximation techniques has been developed, Dingemans (1997). Although the non-linear effects become significant as the shoreline is approached, a consistent linear solution is still very useful, providing a great deal of information concerning the wave field and its impact on the nearshore environment; see, e.g., Massel (1989). In addition, linear theory serves as the starting point for weakly nonlinear models, permitting the calculation of wave energy dissipation due to bottom friction and wave breaking by an indirect process, Dingemanns (1997), Massel & Brinkman (2001).

One very attractive family of models in variable bathymetry is obtained by reformulating the wave propagation problem as a system of equations on the horizontal plane with variable coefficients. Berkoff (1972) derived an one-equation model for gentle bottom slopes, called the mild-slope equation, in which the vertical distribution of the wave potential has been prescribed. Other derivations of similar or improved one-equation models, using either averaging techniques or variational principles, have been given by various authors; see e.g. the general survey by Porter & Chamberlain (1997). The basic restriction inherent to any one-equation model is that the vertical structure of the wave field is given by a specific, preselected function. In order to better describe the wave field when the bottom topography is not slowly varying and the depth is sufficiently small so that the wave strongly interacts with the bottom, Massel (1993) and Porter & Staziker (1995) derived extended mild-slope models, in which the vertical profile of the wave potential at any horizontal position is represented by a local-mode series involving the propagating and the evanescent modes. However, this expansion has been found to be inconsistent with the Neumann boundary condition on a sloping bottom, since each of the vertical modes involved in the local-mode series violates it and, thus, the solution, being a linear superposition of modes, behaves the same. This fact has two important consequences. First, the velocity field in the vicinity of the bottom is poorly represented and, secondly, wave energy is not generally conserved. This problem has been remedied by the consistent coupled-mode model developed by Athanassoulis & Belibassakis (1999).
In the latter model the standard local-mode representation is enhanced by including an additional term, called the sloping-bottom mode, leading to a consistent coupled-mode system of equations. This model is free of any assumptions concerning the smallness of bottom slope and curvature, and it is consistent since it enables the exact satisfaction of the bottom boundary condition and the correct calculation of the bottom velocities. A key feature of the above coupled-mode technique is that the rate of decay of the modal amplitudes is very fast (see Athanassoulis & Belibassakis 1999), and thus, only and a small number of modes is sufficient for a very accurate calculation. These facts facilitated its extension to treat wave propagation and diffraction in general three-dimensional environments, Belibassakis et al. (2001), and to predict second-order waves in variable bathymetry, Belibassakis & Athanassoulis (2002).

In the present work the consistent coupled-mode model has been further extended to include the effects of bottom friction and wave breaking, which are important for the more accurate calculation of the radiation stresses, especially on decreasing depth. Furthermore, the coupled-mode model has been used in conjunction with an iterative solver of the mean-flow equations, permitting an accurate calculation of wave-induced set-up and flow in intermediate and shallow water environments. Finally, the resulting phase-averaged mean-pressure has been applied to calculate the induced groundwater circulation on a permeable beach, in the set-up region. Under the assumption that the groundwater flow is in the Darcy law regime, in the case of a stationary, phase-averaged flow, the velocities can be obtained in terms of the pressure gradient; see also Massel (2001).

THE WAVE POTENTIAL IN VARIABLE BATHYMETRY

The marine environment consists of a water layer bounded above by the free surface and below by a bottom, separating water from porous medium, Fig.1. The bottom surface exhibits a general variation, i.e. the bathymetry is characterised by parallel, straight bottom contours lying between two regions of different depth: the deeper water region or region of incidence (\( x < a \)), where the depth is constant and equal to \( h = h_1 \)) and the shallower water region (\( x > b \)), where the depth is constant and equal to \( h = h_1 \)). A Cartesian coordinate system \( \{x,y,z\} \) is introduced, with its origin at the intersection between the plane beach topography and the mean water level, and the z-axis pointing upwards. The liquid domain \( D \) is decomposed in three parts \( D^m, m=1,2,3 \) (see Fig. 1), where \( D^{(2)} \) is the variable bathymetry subdomain (\( a < x < b \)). The free-surface \( \partial D_F \) and bottom \( \partial D_{h_1} \) boundaries are similarly decomposed.

The wave field in the region \( D \) is excited by a harmonic incident wave, with direction of propagation normal to the depth-contours. In the framework of linearised water wave equations (Wehausen & Laitone 1960, Mei 1983), the fluid motion is described by the 2D wave potential

\[
\Phi(x,z,t) = \text{Re}\left\{-\frac{igH}{2\mu} \phi(x,z;\mu) \cdot \exp\left(-i\omega t\right)\right\},
\]

where \( H \) is the wave height, \( g \) is the acceleration due to gravity, \( \mu = \omega^2 / g \) is the frequency parameter, and \( i = \sqrt{-1} \). The complex wave potential satisfies the Laplace’s equation on the vertical plane

\[
\frac{\partial^2 \phi(x,z)}{\partial x^2} + \frac{\partial^2 \phi(x,z)}{\partial z^2} = 0, \quad -h(x) < z < 0,
\]

the free-surface boundary condition

\[
\frac{\partial \phi(x,z)}{\partial z} - \mu \phi(x,z) = 0, \quad z = 0,
\]

and the bottom boundary condition

\[
\left( \frac{\partial}{\partial z} + \frac{dh_1}{dx} \frac{\partial}{\partial x} \right) \phi(x,z) = 0, \quad z = -h(x).
\]
The above equations are supplemented by the following conditions at infinity, as \( x \to \pm \infty \),

\[
\varphi(x, z) \approx \left[ e^{i k_0 (z+h_1)} + A_k e^{-i k_0 (z+h_1)} \right] \frac{\cosh\left(k_0^3 (z+h_1)\right)}{\cosh\left(k_0^3 h_1\right)},
\]

(2a)

\[
\varphi(x, z) \approx A_r e^{i k_0 (z+h_1)} \frac{\cosh\left(k_0^3 (z+h_1)\right)}{\cosh\left(k_0^3 h_1\right)},
\]

(2b)

where \( A_k \) and \( A_r \) denotes the reflection and transmission coefficients, and the wavenumbers \( k_0^3, k_0^3 \) are obtained as the positive real root of the dispersion relations, formulated at the depth \( h_1 \) and \( h_2 \), respectively.

\[
\mu h_i = kh_i \tanh (kh_i), \quad i = 1, 3.
\]

(2c)

The free-surface elevation is obtained in terms of the wave potential on the free surface as: \( \eta(x, t) = (-1/g) \partial \Phi(x, z=0, t) / \partial t \). The monochromatic wave propagation-diffraction problem in the variable bathymetry region can be reformulated as a matching boundary value problem on the complex wave potential \( \varphi^2(x, z), (x, z) \in D^2 \). This is obtained by using the following modal-type expansions of the wave potential in the constant-depth half strip \( D^1 \) and \( D^3 \):

\[
\varphi^1(x, z) = \left( \exp(-i k_0^1 x) + A_k \exp(i k_0^1 x) \right) \frac{\cosh\left(k_0^3 (z+h_1)\right)}{\cosh\left(k_0^3 h_1\right)} Z^1_0(z) + \sum_{n=1}^\infty C_n^0 Z_n^1(x) \exp(-k_0^1 (x-a)), (x, z) \in D^1,
\]

(3a)

and

\[
\varphi^3(x, z) = A_r \exp(i k_0^3 x) \frac{\cosh\left(k_0^3 (z+h_1)\right)}{\cosh\left(k_0^3 h_1\right)} Z^1_0(z) + \sum_{n=1}^\infty C_n^3 Z_n^3(x) \exp(k_0^3 (b-x)), (x, z) \in D^3.
\]

(3b)

The first terms \( n=0 \) in the series (3a) and (3b), respectively, are the propagating modes, while the remaining ones \( n=1, 2, \ldots \) are the evanescent modes. In the above expansions the sets of numbers \( \{i k_0^1, k_0^1, n=1, 2, \ldots \}, i = 1, 3 \), and the sets of vertical functions \( \{Z^1_n(z), n=0, 1, 2, \ldots \}, i = 1, 3 \), are the eigenvalues and the corresponding eigenfunctions of regular Sturm-Liouville problems obtained from the Laplace equation (by separation of variables) in the constant-depth half-strips \( D^3, i = 1, 3 \). The eigenvalues are given as the roots of the dispersion relations

\[
\mu h_i = -k_0^i \tanh \left(k_0^i h_i\right), \quad n=0, 1, 2, \ldots, \quad i = 1, 3,
\]

(4a)

and the corresponding eigenfunctions are given by

\[
Z^1_0(z) = \frac{\cosh\left(k_0^1 (z+h_1)\right)}{\cosh\left(k_0^1 h_1\right)}, \quad Z^3_0(z) = \frac{\cosh\left(k_0^3 (z+h_1)\right)}{\cosh\left(k_0^3 h_1\right)}.
\]

(4b)

Then, the coefficients \( A_k, \{C^1_n\}_{n=0}^\infty \) and \( A_r, \{C^3_n\}_{n=0}^\infty \), and the wave potential \( \varphi^2(x, z) \) in \( D^2 \) can be obtained as the solution of the Laplace equation on the vertical plane in \( D^2 \), satisfying the free-surface condition on \( \partial D^2 \), the bottom boundary condition on \( \partial D^2 \), and matching conditions requiring continuity of wave velocity and pressure on the vertical interfaces, \( \partial D^2 \) (\( x = a \)) and \( \partial D^2 \) (\( x = b \)), separating the variable bathymetry region \( D^2 \) from \( D^3 \) and \( D^3 \). The wave potential \( \varphi^2(x, z) \) associated with the propagation/diffraction of water waves in the variable bathymetry region can be very conveniently treated by means of the consistent coupled-mode model (Athanassoulis & Belibassakis 1999). This model is based on the following enhanced local-mode representation:

\[
\varphi^2(x, z) = \varphi^1(x) Z_1(z; x) + \varphi_1(x) Z_0(z; x) + \sum_{n=1}^\infty \varphi_n(x) Z_n(z; x).
\]

(5)

In the above expansion the mode \( n=0 \) \( (\varphi_0(x) Z_0(z; x)) \) denotes the propagating mode and the remaining terms \( n=1, 2, \ldots \), are the evanescent modes. The additional term \( \varphi_1(x) Z_1(z; x) \) is a correction term, called the sloping-bottom mode, which properly accounts for the satisfaction of the Neumann bottom boundary condition on the non-horizontal parts of the bottom. The functions \( Z_n(z; x), n=0, 1, 2, \ldots \), appearing in Eq. (5) are obtained as the eigenfunctions of local vertical Sturm-Liouville problems,

\[
Z_0(z; x) = \frac{\cosh\left(k_0 (z+h)\right)}{\cosh\left(k_0 h\right)}, \quad Z_n(z; x) = \cos\left(k_0 (z+h)\right), \quad n=1, 2, \ldots
\]

(6a)

where the \( x \)-dependent eigenvalues \( \{i k_0, k_0 \} \) are obtained as the roots of the local dispersion relation

\[
\mu h(x) = -k(x) h(x) \tan[k(x) h(x)], \quad a \leq x \leq b.
\]

(6b)

A specific convenient form of the function \( Z_1(z; x) \) is

\[
Z_1(z; x) = h(x) \left[ \frac{z}{h(x)} \right]^3 + \left[ \frac{z}{h(x)} \right]^2,
\]

(7)

however, other choices are also possible; see Athanassoulis & Belibassakis (1999). By introducing the above expansion in a variational principle, the coupled-mode system of horizontal equations for the amplitudes of the wave potential is obtained:

\[
\sum_{n=1}^\infty a_{nm}(x) \varphi_n^m(x) + b_{nm}(x) \varphi_n^m(x) + c_{nm}(x) \varphi_n^m(x) = 0,
\]

(8)

where \( m = -1, 0, 1, \ldots \), in \( a < x < b \), where a prime denotes differentiation with respect to \( x \). The coefficients \( a_{nm}, b_{nm}, c_{nm} \) of the coupled-mode system (8) are dependent on \( x \) through \( h(x) \) and can be found in Table 1 of Athanassoulis & Belibassakis (1999). The coupled-mode system is supplemented by appropriate boundary conditions on the mode amplitudes, at the ends \( x=a \) and \( x=b \) of the variable bathymetry region, ensuring complete matching on \( \partial D^2 \) and \( \partial D^2 \). Finally, from the solution of the coupled-mode system, the reflection and transmission coefficients can be calculated; see Athanassoulis & Belibassakis (1999). An important feature of the
calculation of the wave field by means of the enhanced representation (5) is that it exhibits an improved convergence, since the rate of decay of the modal amplitudes $|\phi_n|$ of the order $O(n^{-1})$. Thus, only a few number of modes suffice to obtain a convergent solution in this region, even for bottom slopes of the order of 1:1, or higher. More details about the above technique can be found in Athanassoulis & Belibassakis (1999), and as concerns its application to realistic 3D seabed topographies in Belibassakis et al (2001). Moreover, details about the extension of the above theory to second order Stokes waves in variable bathymetry can be found in Belibassakis & Athanassoulis (2002).

WAVE ENERGY DISSIPATION

The coupled-mode system in the form of Eq. (8) deals with ideal (inviscid) flow. However, the dissipation of wave energy due to bottom friction and wave breaking can be indirectly taken into account, by including an appropriate imaginary part in the wave number, similarly as in the case of the mild-slope and the extended mild-slope models; see, e.g., Massel (1992), Dingemans (1997). Taking into account that the wave propagation features are essentially described by the propagating mode $\phi_+(x)$, in order to include energy dissipation effects in the present coupled-mode system, the latter is modified as follows

$$\sum_{n=1}^{\infty} a_{mn}(x) \phi_m^*(x) + b_{mn}(x) \phi_n^*(x) + (c_{mn}(x) + i\gamma k_0 \delta_m) \phi_n(x) = 0,$$

$$m = -1, 0, 1, \ldots, \quad (9)$$

where $\delta_m$ is Kronecker’s delta and $\gamma$ denotes the dissipation coefficient (Massel 1992),

$$\gamma(x) = \gamma_f(x) + \gamma_b(x),$$

combining the effects of seabed friction and wave breaking on wave propagation. The bottom friction dissipation coefficient $\gamma_f(x)$ is obtained in terms of the local wave-velocity at the bottom, as follows (Massel & Gourlay, 2000),

$$\gamma_f(x) = \frac{16 f_u}{3\pi} \left| u_b(x) \right|^2 \frac{1}{g C_g(x) H(x)},$$

where $u_b(x)$ denotes the bottom orbital velocity, and $H(x)$ is the local wave height. In Eq. (11), $C_g$ is the local wave group velocity, and for turbulent flow, the bottom friction coefficient $f_u$ is dependent on the bottom surface roughness, taking values in the interval $0.1 < f_u < 0.2$ in natural conditions, Nelson (1996). The latter is of order of 0.01 for sandy bottoms, Longuet-Higgins (1970); see also Dingemans (1997). Following Massel & Gourlay (2000), the wave breaking dissipation coefficient $\gamma_b(x)$ is modelled as

$$\gamma_b(x) = \frac{\alpha \omega}{\pi} \sqrt{g h(x) H(x)} C_g(x) C_g h(x),$$

when $H(x) \frac{H_u(x)}{h(x)} > H_u(x) h(x)$, \quad (12)

and zero otherwise. In the above equation $C = \alpha \omega/k$ is the local phase velocity, $H_u(x)$ is the local maximum allowable wave height, and $\alpha = O(1)$ is an empirical coefficient. In the present work, the expressions suggested by Massel & Gourlay (2000) have been used for $H_u$ and $\alpha$. In particular, the following formula has been used to determine the subregions where breaking occurs,

$$H_u(x) = \left\{ \begin{array}{ll} \frac{0.937}{1 + 0.125} \left( \frac{H}{\lambda} \right)^{0.130}, & \text{when } \frac{dh}{dx} > 0.025 \\ \frac{-1 + \sqrt{1 + 0.01504 h_{*}^{2.5}}}{0.01654 h_{*}^{-1.25}}, & \text{when } \frac{dh}{dx} \leq 0.025 \end{array} \right.,$$

where $h_* = h / g T^2$ and $\lambda, T$ denote the incident wave length and period, respectively. Finally, as concerns the empirical coefficient $\alpha$, this is taken to be given in terms of the bottom slope $h'$ and the non-linearity parameter $F_{ob} = (H / h_{*ob})^{1/2} (T \sqrt{g / h_{*ob}})^{5/2}$ (based on the incident wave height $H$) by the following formula,

$$\alpha(h', F_{ob}) = (0.45 + 0.1h') \sqrt{(F_{ob} - 100)/380}, \quad \text{when } F_{ob} > 100,$$

and zero otherwise. The latter has been found to accurately reproduce various data in cases with moderate and large bottom slopes examined by Massel & Gourlay (2000).

MEAN FLOW EQUATIONS IN VARIABLE DEPTH

Shoaling, refraction, diffraction and dissipation processes, in intermediate and shallow water depth, induce changes in the radiation stresses, which act as specific type of forcing to the depth-averaged momentum equations and produce variations to the mean sea level and the mean flow. The purpose of this section is to present and discuss the set of equations that will be used, in conjunction with the coupled-mode model (9), for calculating the wave-induced mean flow and set-up in variable bathymetry regions. The averaged or mean-flow equations are derived from the continuity and the momentum (Euler) equations for incompressible flow, subjected to the kinematic and dynamic boundary conditions; see, e.g., Mei (1983, Sec. 10.2) or Dingemans (1997, Sec. 2.9). The derivation is based on the separation of the fast-scale from the slow-scale of the flow motion and vertical averaging. Introducing the following definitions

$$u(x, z; t) = U(x,t) + \bar{u}(x, z; t), \quad w(x, z; t) = W(x,t) + \bar{w}(x, z; t),$$

where

$$U(x,t) = \frac{1}{h + \eta_h} \left\{ \int_{z-h}^{0} u(x,z; t) dz \right\}, \quad W(x,t) = \frac{1}{h + \eta_h} \left\{ \int_{z-h}^{0} w(x,z; t) dz \right\},$$

and similarly for the free-surface elevation,

$$\eta(x,t) = \eta_h(x,t) + \bar{\eta}(x,t), \quad \bar{\eta}(x,t) = 0,$$

where the operator $\{ \cdot \} \int_{z-h}^{0} \int_{t-\delta T}^{t} \cdots dt$ denotes time averaging in the fast time scale (in the time-interval $\delta T$). The system of mean flow equations consists of the depth-averaged continuity equation and the conservation of depth-averaged momentum. The continuity equation (mass conservation) leads to:

$$\frac{\partial \eta_h}{\partial t} + \frac{\partial}{\partial x} \left( (h + \eta_h) U \right) = 0,$$

which, in the case of steady mean flow results in the following form

$$\frac{\partial}{\partial x} \left( (h + \eta_h) U \right) = 0.$$
The latter can be integrated to give the distribution of the mean flow in terms of the set-up \( \eta_m(x) \), as follows
\[
U(x) = \frac{U_0}{h(x) + \eta_m(x)},
\]
(16c)
where \( U_0 \) is a constant controlling the incoming mean flow. The conservation of depth-averaged momentum in its time-independent version reads as follows, (Dingemans, 1997),
\[
\rho(h + \eta_m)[U \frac{\partial U}{\partial x} + g \frac{\partial \eta_m}{\partial x}] + \frac{\partial S_{SS}}{\partial x} \approx \{\tau^s\} - \{\tau^s\} + \frac{dh}{dx} \left( \frac{\partial}{\partial x} \int_{-\eta_m}^{0} \rho \vec{u} dz \right),
\]
(17)
where \( \{\tau^s\}, \{\tau^s\} \), denote the average free-surface and bottom-surface stresses. The term \( S_{SS} \) is the radiation stress, defined by
\[
S_{SS}(x) = \left( \int_{-h(x)}^{0} (\rho \vec{u}^2) dz \right) + \left( \int_{-h(x)}^{0} p dz \right) - \frac{1}{2} \rho g \left( h + \eta_m \right)^2.
\]
(18)
For relatively short fetches or without input of energy by wind, the free-surface stress is considered to be considerably smaller than the bottom stress, \( \{\tau^s\} \ll \{\tau^s\} \), and thus, it is usually neglected.

Moreover, in the case of a mildly-sloped bottom the last term in the right-hand side of Eq. (17) can be dropped, resulting in the following, simplified form of the averaged momentum equation
\[
\rho(h + \eta_m)[U \frac{\partial U}{\partial x} + g \frac{\partial \eta_m}{\partial x}] = -\frac{\partial S_{SS}}{\partial x} - \{\tau^s\}.
\]
(22)

The leading-order term of the radiation stress is obtained from Eq. (18) as follows
\[
S_{SS}(x) = \left( \int_{-h(x)}^{0} (\rho \vec{u}^2) dz \right) + \left( \int_{-h(x)}^{0} p dz \right) - \frac{1}{2} \rho g \left( h + \eta_m \right)^2,
\]
(23)
where \( \vec{u}(x,z;t) = \partial \Phi(x,z,t) / \partial x \). Moreover, in the case when the complex wave field \( \Phi(x,z) \) can be approximated by keeping only the propagating mode in the local-mode series (5),
\[
\Phi(x,z,t) = \phi_m(x) \frac{\cosh \left( k(z+h) \right)}{\cosh (kh)} \exp(-i\omega t),
\]
then the expression of the radiation stress is further simplified as follows (see also Dingemans, 1997, Sec. 1.9):
\[
S_{SS}(x) = \frac{H^2 \delta S_{SS}(x)}{\rho \omega^2},
\]
(24)
where
\[
\delta S_{SS}(x) = 2CC_s \left[ \frac{\partial \phi_m}{\partial x} \right] + \omega^2 \left( \frac{2C_s}{C} \right) \frac{\partial \phi_m}{\partial x} + \frac{1}{2} (gh - CC_s) \frac{\partial \phi_m}{\partial x}^2.
\]

The bottom stress \( \{\tau^s\}\), which together with the horizontal gradient of radiation stress \( S_{SS} \) constitute the forcing terms of the averaged momentum equation, can be modeled in terms of a frictional coefficient \( c_f \). For the flow under the wave motion a linear formulation based on the bottom wave velocity is
\[
\{\tau^s\} = \frac{1}{2} c_f \rho |u_b| U.
\]
(25)

Eq. (22) can be further partially linearised, Massel & Brinman (2001), obtaining the following, simplified form of the averaged momentum equation
\[
(h + \eta_m) \frac{\partial \eta_m}{\partial x} = -\frac{1}{\rho g} \frac{\partial S_{SS}}{\partial x} - \frac{1}{2g} c_f |u_b| U,
\]
(26)
where \( c_f \) is a (globally average) value of the bottom friction coefficient, and \( |u_b| \) is a representative value of the bottom wave-induced orbital velocity. Eliminating the mean velocity \( U \) from Eqs. (16b) and (26) we obtain the following second-order equation
\[
\frac{d^2 \eta_m}{dx^2} + \left[ \frac{2}{h + \eta_m} \frac{dh}{dx} \frac{F_i}{h + \eta_m} \right] \frac{dh}{dx} \frac{1}{r} \frac{dr}{dx} \frac{d\eta_m}{dx} - \frac{1}{h + \eta_m} \left[ \frac{dF_i}{dx} + \frac{F_i}{h + \eta_m} \frac{dh}{dx} \frac{F_i}{r} \frac{dr}{dx} \right] = 0,
\]
(27)
in which \( F_i = -(\rho g)^{-1} dS_{SS} / dx \) and \( r = c_f |u_b| / 2g \). The only additional approximation made in Eq.(27) is the assumption that \( (d\eta_m / dx)^2 \approx 0 \). Furthermore, using the following approximation \( h + \eta_m \approx h \), then we obtain that \( r = \text{const} \) and \( F_i / (h + \eta_m)^2 \approx 0 \).

In that case, the equation (27) finally reduces to the linear model derived by Massel and Brinkman (2001),
\[
\frac{d^2 \eta_m}{dx^2} + \frac{2}{h} \frac{d\eta_m}{dx} \frac{1}{h} \frac{dF_i}{dx} = 0.
\]
(28)
We note here that for water depth \( h(x) \) changing linearly with the horizontal distance, Eq. (28) can be analytically solved (Massel and Brinkman, 2000), leading to useful results, especially for the understanding of wave induced set-up and mean flow characteristics on shoals that can be approximated by piecewise linear functions.

**Boundary conditions**

The present model is based on the system of Eqs. (16b) and (22), in conjunction with Eq. (18) (or 23) for the calculation of the radiation stress and Eqs. (25) for the bottom stress. In the case of **open domains**, where mean flow is permitted, the system is solved as a boundary value problem, based on the assumption of zero set-up far from the bottom inhomogeneity
\[
\eta_m(x \to -\infty) = 0, \quad \eta_m(x \to \infty) = 0.
\]
(29)

This represents also the case of Gourlay(1996) open-lagoon experiments where the water level in front of the step (“ocean”) and behind the step (“lagoon”) was kept a constant. In the case of **closed domains** mean flow is necessarily zero \( U = 0 \), and the present system finally reduces to a first-order equation
\[
(h + \eta_m) \frac{\partial \eta_m}{\partial x} = -\frac{1}{\rho g} \frac{\partial S_{SS}}{\partial x},
\]
(30)
which can be solved as an initial value problem. In this case, only
one boundary condition is needed for the integration of Eq. (30) and an appropriate one is the following
\[ \eta_\infty (x \to -\infty) = 0, \quad (31) \]
justified by the assumption of zero set-up in the deeper water region.

NUMERICAL RESULTS

Extended numerical predictions obtained by the present model, in closed and open domains (where also flow is permitted), including detailed comparisons with experimental data, will be presented elsewhere. As an example, we present in Figure 2 results obtained by the present system concerning the wave height distribution, the set-up and the mean flow over an idealised reef profile with very steep bottom topography (bottom slope 100%). The case considered corresponds to an open domain, where also flow is permitted, and it has been studied experimentally by Gourlay (1996). The results shown in Fig.2 are for an incident wave period \( T = 1.1 \text{s} \) and wave height \( H = 1.31 \text{m} \). The measured values of the max wave set-up and mean flow in this case are \( \eta_{\text{max}} = 7 \text{mm} \) and \( U_{\text{max}} = 37.3 \text{cm/s} \), and agree relatively well with the corresponding values as predicted by the present model, which are 8mm and 33cm/s, respectively.

![Figure 2](image2.png)

**Figure 2.** Wave height distribution, set-up and mean flow over an idealised reef profile (100% face slope). Incident wave period \( T = 1.1 \text{s} \), and incident wave height \( H = 1.31 \text{m} \).

![Figure 3](image3.png)

**Figure 3.** Numerical results obtained in the case of an open domain shown in (a), as obtained by (b) the present coupled-mode system, in conjunction with the nonlinear mean flow equations Eqs. (16, 26), (b) the simplified model based on Eqs. (16, 27), and (c) the model (28), developed by Massel & Brinkman (2001). The incident wave conditions are \( H = 2 \text{m} \) and \( T = 4 \text{sec} \). The horizontal variation of the local wave height is also presented in (a), as obtained by the present coupled-mode system, Eqs. (9).
Next, we focus on the comparison between the numerical results obtained by the present method, based on: (a) the complete model based on the nonlinear mean flow equations (16b) and (26), (b) the simplified model defined by Eq. (27), and (c) the same model after linearization and additional approximations given by Eq.(28) which is developed by Massel & Brinkman (2001). We consider the case of the open domain shown in Fig. 3(a), which is also examined by the last authors. The shoal face has slope 1:25. In the examined case of mildly-sloped bottom topography the predictions by all models are found to be in good agreement as concerns the general mean-flow characteristics. However, the simplified models (27) and (28) tend to overestimate the cross-shoal transport by ~15% in comparison with the results provided by the present nonlinear system based on Eqs. (16b) and (26).

Finally, the present model has been applied to calculation of groundwater flow in porous sandy beaches, with application to the prediction of steady groundwater flow due to wave set-up. Assuming that the groundwater flow is in the Darcy law regime, and in the case of a stationary mean flow, the velocities in the porous medium can be obtained in terms of the pressure gradient as follows

\[ \mathbf{u} = -k \nabla p, \]  

where \( k \) denotes the coefficient of hydraulic conductivity divided by the specific weight of water; see, e.g., Massel (2001).

For a homogeneous porous medium, the problem concerning groundwater circulation is governed by the Laplace’s equation on the pressure,

\[ \frac{\partial^2 p(x,z)}{\partial x^2} + \frac{\partial^2 p(x,z)}{\partial z^2} = 0, \quad -\infty < z < -h(x), \]

forced by Dirichlet data on \( z = -h(x) \), specified by the excess pressure on the sea bottom surface. The latter is calculated from the hydrostatic pressure associated with the mean-flow set-up. In Fig. 4, we present the application of the present model to the calculation of wave flow and wave set-up in a shallow sloping beach of 2% slope. The incident wave conditions are: \( H=1m, T=6sec \). In order to treat the wedge singularity associated with the wave potential, we use a cut-off depth \( (h_c=0.15m) \) and calculate the wave field from the solution of the coupled-mode system, Eqs. (9). The calculated wave set-up in the region is shown in Fig. 4, as obtained from the mean flow equations (16b) and (26). On the basis of the latter result, the excess hydrostatic pressure distribution on the bottom surface of the

Figure 4. Wave height distribution and set-up in a shallow sloping beach (2% slope).

Figure 5. Excess pressure distribution on the bottom surface of the shallow sloping beach of Fig.4, induced from the wave set-up in the wedge region.

Figure 6. Calculated porous flow field

Figure 7. Streamlines of the calculated groundwater flow.
wedge-like shallow sloping beach is obtained by cutting off (using a Gaussian filter) the shoreward part of the wave set-up after the point corresponding to the initial position of the coastline (apex of the wedge), Fig. 5. Then, the porous flow potential is calculated as a solution to the Laplace equation for the excess pressure in the porous sediment, with Dirichlet boundary condition on the bottom surface, specified by the calculated excess hydrostatic pressure distribution on the bottom surface. The numerical solution of the above boundary value problem is obtained by direct application of the Boundary Element Method, using elementary (Rankine) sources distribution. From the porous potential the corresponding groundwater flow is obtained by differentiation, using a coefficient of permeability derived though the grain diameter \( D_{50} \) by using an approximate version of Hazen formula; see, e.g., Massel (2001, Eq.6). In the present example the groundwater flow field is shown in Fig. 6. The corresponding streamlines are calculated by direct numerical integration of the velocity field, and are shown in Fig. 7.

CONCLUSIONS

A complete, phase-resolving wave model is coupled with an iterative solver of the nonlinear mean-flow equations, permitting an accurate calculation of wave set-up and cross shoal transport in intermediate and shallow-water (open or closed) domains, characterised by possibly steep bathymetric variations. The present wave model is based on the consistent coupled-mode theory, developed by Athanassoulis & Belibassakis (1999) for the calculation of wave field over variable bathymetry regions, which has been recently extended to 3D by Belibassakis et al (2001) and to second-order Stokes waves in variable bathymetry regions by Belibassakis & Athanassoulis (2002). The latter wave model has been further enhanced to include the effects of bottom friction and wave breaking, which are important for the correct calculation of radiation stresses in the nearshore and onshore region. Subsequently, the present wave model has been coupled to the mean flow equations enabling us to obtain enhanced predictions of wave-induced set up and mean flow characteristics in general domains. Comparisons in various cases has shown that the present model provides results in compatibility with other (more simplified) theoretical models, without adding significantly into the computational cost. Finally, the resulting phase-averaged mean-pressure has been used to calculate the induced groundwater circulation in a permeable beach, in the set-up region. The present approach is expected to find useful applications to the estimation the effects of the wave field on shoals and reefs, and its impact on the nearshore/coastal environment.

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