A weakly nonlinear, coupled-mode model is developed for the wave-current-seabed interaction problem, with application to wave scattering by steady currents over general bottom topography. Based on previous work by the authors (Athanassoulis & Belibassakis 1999, Belibassakis et al 2001), the vertical distribution of the scattered wave potential is represented by a series of local vertical modes containing the propagating mode and all evanescent modes, plus an additional term accounting for the bottom boundary condition when the bottom slope is not negligible. Using the above representation, in conjunction with Luke's (1967) variational principle, the wave-current-seabed interaction problem is reduced to a coupled system of differential equations on the horizontal plane. If only the propagating mode is retained in the vertical expansion of the wave potential, and after simplifications, the present system is reduced to an one-equation model compatible with Kirby's (1984) mild-slope model with application to the problem of wave-current interaction over slowly varying topography. The present coupled-mode system is discretized on the horizontal plane by using a second-order finite difference scheme and numerically solved by iterations. Numerical results are presented for two representative test cases, demonstrating the importance of the first evanescent modes and the sloping-bottom mode. The analytical structure of the present model facilitates its extension to treat fully non-linear waves, and it can be further elaborated to study wave propagation over random bottom topography and general currents.

1. INTRODUCTION

The evolution of water-waves in nearshore and coastal areas, and especially, in regions where ambient, tidal and other type of currents co-exist, is important for a variety of engineering applications including interaction of waves with structures, coastal management, harbour maintenance, exploitation of renewable energy sources etc. Sometimes the effects of currents on wave transformation can be substantial, since they are responsible for creating Doppler shift and causing wave refraction, reflection, and breaking. Also, currents could completely change the wave energy pattern. In particular, the characteristics of surface waves present significant variation as they propagate through non-homogeneous ambient currents, in the presence of depth inhomogeneities in variable bathymetry regions. Thus, large amplitude waves can be produced when obliquely propagating waves interact with opposing currents, see, e.g., Mei (1983, Ch.3.7). This situation could be further enhanced by inshore effects due to sloping seabeds, and has been reported to be connected with the appearance of “giant waves”, Lavrenov & Porubov (2006). Extensive reviews on the subject of wave-current interaction in the nearshore region have been presented by Peregrine (1976), Jonsson (1990) and Thomas & Klopman (1997).

The study of spatial evolution of water waves and the investigation of scattering of realistic wave spectra over irregular currents, with characteristic length of variation comparable to the dominant wavelength, and including the effects of bottom irregularities, can be supported by theoretical models treating the simpler problem of monochromatic waves interacting with steady inhomogeneous currents. Wave-current interaction models over slowly varying topography have been developed and studied by various authors. Under the assumption of irrotational wave motion, Kirby (1984) derived a phase-resolving one-equation model, generalizing the Berkhoff’s mild-slope equation in regions with slowly varying depth and ambient currents; see also Liu (1990). The latter
model in its elliptic time-harmonic form has been exploited, in conjunction with numerical (finite-element, finite difference etc) solvers, to numerous wave-current-seabed interaction applications, Chen et al (2005).

In the present work, a new, weakly nonlinear, coupled-mode model is developed for the wave-current-seabed interaction problem, with application to wave scattering by steady ambient currents over general bottom topography. Based on previous work by the authors (Athanassoulis & Belibassakis 1999, Belibassakis et al 2001, Gerostathis et al 2008), the vertical distribution of the scattered wave potential is represented by a series of local vertical modes containing the propagating mode and all evanescent modes, plus an additional term accounting for the bottom boundary condition when the bottom slope is not negligible (the sloping-bottom mode). Using the above representation, in conjunction with Luke’s (1967) variational principle, the wave-current-seabed interaction problem is reduced to a coupled system of differential equations on the horizontal plane. If only the propagating mode is retained in the vertical expansion of the wave potential, and after simplifications and linearisation, the present system is reduced to a one-equation model (the wave-current modified mild-slope model), which after additional simplifications is shown to be compatible with Kirby’s (1984) mild-slope model. The present coupled-mode system is discretized on the horizontal plane by using a second-order finite difference scheme and numerically solved by iterations. Results are presented for two representative test cases corresponding to a rip current over a sloping beach in shallow water, and to a ring current in constant depth. In these cases comparisons are presented of the results obtained by the one-equation model (i.e. keeping only the propagating mode in the expansion of the wave potential) and the more complete expansion of the wave potential (including the rest of the vertical modes), demonstrating the importance of the first evanescent modes and the sloping-bottom mode. The analytical structure of the present model facilitates its extension to treat fully non-linear waves, and it can be further elaborated to study wave propagation over random bottom topography and scattering by currents with general 3D structure.

2. THE BACKGROUND CURRENT FLOW

We consider wave propagation, in the presence of ambient, non-homogeneous current, in a variable bathymetry region, Fig. 1. The liquid is assumed inviscid and homogeneous, and the flow associated with the background current is assumed to be self-existent, steady and possibly weakly rotational. On the other hand, the wave flow perturbing the background current flow, generated by an incident wave system coming from the far up-wave region (see Fig.1), is assumed to be irrotational. The background current flow is assumed to be nearly horizontal. Moreover, its velocity is assumed to be small and slowly varying, and thus, the associated mean free-surface elevation (set-down) is also small. We introduce however, no assumption as regards the mildness of the bottom slope.

A Cartesian coordinate system is used, having its origin at some point on the unperturbed free-surface \((z = 0)\). The \(z\)-axis is pointing upwards and one of the horizontal axes is taken to be (approximately) aligned with the mean direction of the transmitted wave field; see Fig. 1. The current \(\mathbf{q} = (U_1, U_2, W)\) has been assumed to be steady and self existent and the kinematics of this flow require that

\[
\nabla \cdot \mathbf{q} = 0, \tag{1a}
\]

\[
\mathbf{q} \cdot \mathbf{n} = 0, \quad z = H(x_1, x_2), \quad \mathbf{q} \cdot \mathbf{n} = 0, \quad z = -h(x_1, x_2), \tag{2b,c}
\]

where \(\nabla\) denotes the gradient operator in 3D, and \(H\) denotes the mean set-down associated with the background current flow. Eq. (2.c) is equivalently written in the form

\[
W + U_1 \frac{\partial h}{\partial x_1} + U_2 \frac{\partial h}{\partial x_2} = 0, \quad z = -h(x_1, x_2), \tag{2c'}
\]

Also, using the fact that the current velocity has been assumed to be small, the mean set-down can be neglected \((H \approx 0)\), and the kinematical free-surface boundary condition (2b) can be linearised as follows:
\[ W = 0, \quad z = 0. \]  

The dynamics of the steady background current are described by the corresponding Bernoulli equation, stating that total energy is conserved along the streamlines (Bachelor 1967),

\[ Q = \frac{1}{2} \left[ q^2 \right] + \frac{p}{\rho} + gz = \text{const} = \frac{1}{2} \left[ q^2 \right] + \frac{p}{\rho} + gz_x, \]  

where \( \rho \) is the liquid density, \( g \) is the gravitational acceleration and \( z_x \) denotes the vertical position of each streamline at infinity. Taking \( P_s \) to be the static pressure of the fluid at rest, obtained by the superposition of the atmospheric (\( P_a \)) and the hydrostatic (\( -\rho gz_x \)) pressure at infinity,

\[ P_s = P_a - \rho gz_x, \]  

we finally obtain the following equation

\[ \frac{P}{\rho} = \frac{1}{2} \left[ \left| q \right|^2 \right] + \frac{P_s - gz}{\rho}. \]  

The total (current and wave) flow \( u \), as well as the background current flow \( q \), both satisfy Euler equations. Subtracting these equations by parts and omitting approximately the terms associated with the vorticity of the background flow \( (\nabla \times q) \), which is assumed to be weak, we finally obtain the following approximate Bernoulli equation

\[ \frac{p}{\rho} + \frac{\partial \phi}{\partial t} + q \cdot \nabla \phi + \frac{1}{2} \left| \nabla \phi \right|^2 = 0, \]  

where \( \phi(x, x_z, z; t) \) denotes the wave potential,

\[ \nabla \phi = u - q, \]  

and \( p \) is the wave-disturbance pressure, defined as the difference between the total (\( P_s \)) and the background (\( P \)) pressure,

\[ p = P_s - P. \]  

Using the above equations, we finally obtain the following Bernoulli equation expressing the conservation of energy of the studied wave-current problem:

\[ \frac{\partial \phi}{\partial t} + q \cdot \nabla \phi + \frac{1}{2} \left| \nabla \phi \right|^2 = 0, \]  

where \( \eta(x, x_z; t) \) denotes the free-surface elevation associated with the total wave and current flow. Using Eq. (9), the above equation is written as

\[ \mathcal{L} = \rho \int_{\eta(x, x_z; t)}^{z=\eta(x, x_z; t)} \left[ \frac{g}{\partial t} + \frac{\partial \phi}{\partial t} + \frac{1}{2} \left| u \right|^2 - \left| q \right|^2 \right] dx_z dx_t dt. \]  

Using the fact that the current has been assumed essentially horizontal \( (W \ll U_1, U_z) \) and slowly varying \( \left( \left| q \right|^2 \approx \left| q_0 \right|^2 \right) \), the quantity \( \left| u \right|^2 - \left| q \right|^2 \) can be approximated as follows:

\[ \left| u \right|^2 - \left| q \right|^2 \approx U_1 \frac{\partial \phi}{\partial x_1} + U_z \frac{\partial \phi}{\partial x_2} + \left( \nabla \phi \right)^2. \]  

Requiring the above functional to be stationary, \( \delta \mathcal{L} = 0 \), and after carrying out the algebra, we finally obtain the following equation (see also Massel 1989, Eq.1-35)

\[ \int_{\eta(x, x_z; t)}^{z=\eta(x, x_z; t)} \left[ \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \phi}{\partial x_1} + U_z \frac{\partial \phi}{\partial x_2} + \left( \nabla \phi \right)^2 \right] dx_z dt = 0, \]  

where also the continuity equation concerning the background current, Eq.(1), and the bottom boundary condition, Eq. (2c), have been used. The equations governing the fully non-linear wave-current problem are derived from the above principle.

4. THE NONLINEAR COUPLED-MODE SYSTEM

Following previous works (Athanassoulis & Belibassakis 2002), we introduce the following local-mode series expansion of the wave potential

\[ \phi(x, z; t) = \phi_2(x, t)Z_2(z; h, \eta) + \phi_1(x, t)Z_1(z; h, \eta) + \phi_0(x, t)Z_0(z; h, \eta) + \sum_{n=1}^{\infty} \phi_n(x, t)Z_n(z; h, \eta), \]  

in the variable bathymetry region, where \( x = (x, x_z) \). In the above expansion, the mode \( n = 0 \) (i.e., \( \phi_0(x)Z_0(z; h, \eta) \)) denotes the propagating mode and the remaining terms, \( n = 1, 2, \ldots \), are the evanescent modes. The term \( \phi_1(x, t)Z_1(z; h, \eta) \) is a correction term, called the sloping-bottom mode, properly accounting for the satisfaction of the Neumann bottom boundary condition on the non-horizontal parts of the bottom. The term \( \phi_n(x, t)Z_n(z; h, \eta) \) is an extra term introduced to ensure the satisfaction of the free-surface boundary condition.

In the present case, the functions \( Z_n(z; x), \; n = 0, 1, 2, \ldots \), appearing in Eq. (12), are obtained as the eigenfunctions of...
local vertical Sturm-Liouville problems formulated with respect to the local depth and intrinsic frequency $\sigma = \omega - \mathbf{U} \cdot \mathbf{k}$, as follows,

$$Z_n^* + k_n^2 Z_n = 0, \quad -h(x) < z < 0,$$

$$Z_n'(-h(x)) = 0, \quad z = -h(x), \quad (13a)$$

$$Z_n' - \mu Z_n(0) = 0, \quad z = \eta(x,t) \quad (13b)$$

where the prime denotes differentiation with respect to $z$. The parameter $\mu$ is taken to be the intrinsic frequency parameter, $\mu = \sigma^2 / g$, where $\omega$ denotes a characteristic absolute wave frequency. The definition of the intrinsic frequency depends also on the vector wavenumber $\mathbf{k}$, a quantity which is associated with the wave kinematics (see e.g. Jonsson (1990)), and thus, it is clearly dependent on the solution $\varphi(x_1,x_2,z)$. This fact introduces intrinsic nonlinearity to the wave-current scattering problem, and iterations are necessary for its solution. The local vertical eigenfunctions are obtained from Eqs. (13) as

$$Z_0 = \frac{\cosh[k_n(z+h)]}{\cosh[k_n(\eta+h)]}, \quad Z_n = \frac{\cos[k_n(z+h)]}{\cos[k_n(\eta+h)]}, \quad n = 1,2, \ldots, \quad (14)$$

where the wavenumbers $k_n = k_n(h,\eta)$ are obtained as a solution to the local dispersion relation,

$$\sigma^2 = k_n g \tan[k_n(\eta+h)] = -k_n g \tan[k_n(\eta+h)], \quad n \geq 1. \quad (15)$$

As concerns the sloping-bottom mode $\varphi_{1} Z_{1}$, a specific convenient form of the function $Z_{1}(z;\mathbf{x})$ is given by

$$Z_1(z;\mathbf{x}) = \frac{\mu_1 h_1 - 1}{2 h_1 (\eta+h)} (z+h)^2 - \frac{\mu_1 h_1 + 1}{2 h_1} (\eta+h) + 1, \quad (16)$$

where the parameters $\mu_1, h_1 > 0$ are constant positive, having the following properties: $Z_1'(z = \eta) - \mu_1 Z_1(z = \eta) = 0$, $Z_1'(z = h) = 1/\mu_0$. Finally, as concerns the free surface mode $\varphi_{11} Z_{11}$, a specific convenient form of the function $Z_{11}(z;\mathbf{x})$ is given by

$$Z_{11}(z;\mathbf{x},\mathbf{\theta}) = \frac{\mu_1 h_1 - 1}{2 (\eta+h) h_1} (z+h)^2 - \frac{\mu_1 h_1 + 1}{2 h_1} (\eta+h) + 1, \quad (17)$$

having the properties: $Z_1'(z = \eta) - \mu_1 Z_1'(z = \eta) = 1/\mu_0$, $Z_1'(z = h) = 0$, however other forms are also possible; see also Belibassakis & Athanassoulis (2006) for more details.

Using the representation (12) in the variational equation (11), and after the algebra, we obtain an equivalent reformulation of the wave-current scattering problem in the form of the following nonlinear Coupled Mode System (CMS) of equations,

$$\sum_{n=2}^{\infty} \left[ A_{mn}(\eta) \nabla^2 \varphi_n + B_{mn}(\eta) \nabla \varphi_n + C_{mn}(\eta) \varphi_n \right] +$$

$$+ \frac{\partial \eta}{\partial t} + \mathbf{U} \cdot \nabla \eta - W = 0,$$

$$(18a)$$

$$\sum_{n=2}^{\infty} \left( \frac{\partial \varphi_n}{\partial t} + \left[ Q_n \right] \varphi_n \frac{\partial \eta}{\partial t} + \mathbf{U} \cdot \nabla \varphi_n + W \varphi_n \frac{\partial Z_n}{\partial z} \right) +$$

$$- \sum_{n=2}^{\infty} \left( a_{0,n}^{(0,2)}(\eta) \varphi_n \frac{\partial^2 \varphi_n}{\partial x^2} + a_{1,n}^{(1)}(\eta) \left( \frac{\partial \varphi_n}{\partial x} \frac{\partial \varphi_n}{\partial x} + \frac{\partial \varphi_n}{\partial z} \frac{\partial \varphi_n}{\partial z} \right) +$$

$$+ b_{1,n}(\eta) \varphi_n \frac{\partial \varphi_n}{\partial x} + c_{1,n}(\eta) \varphi_n \varphi_n \right) g \eta + \frac{1}{2} \left( U_{1}^2 - U_{2}^2 + W^2 \right) = 0$$

$$(18b)$$

The matrix coefficients $A_{mn}(\eta)$, $B_{mn}(\eta)$, $C_{mn}(\eta)$ are dependent on the free-surface elevation (\eta) and are given by

$$A_{mn}(\eta) = \langle Z_n, Z_m \rangle = \int_{z=-\eta(x_1)}^{\infty} Z_n(z;\eta) Z_m(z;\eta) dz, \quad (19a)$$

$$B_{mn}(\eta) = 2 \langle \nabla Z_n, Z_m \rangle + \left[ \nabla Z_n, \nabla h + \frac{\partial Z_n}{\partial z} \right] Z_m \right|_{z=-\eta(x_1)}^{\infty}, \quad (19b)$$

$$C_{mn}(\eta) = \langle \frac{\partial^2 Z_n}{\partial z^2}, Z_m \rangle + \left[ \nabla Z_n, \nabla h + \nabla \frac{\partial Z_n}{\partial z} \right] Z_m \right|_{z=-\eta(x_1)}^{\infty} + \left[ \nabla Z_n, \nabla h - \frac{\partial Z_n}{\partial z} \right] Z_m \right|_{z=0(x_1)}^{\infty}, \quad (19c)$$

and the brackets denote the inner product in the vertical interval: $\langle f(z), g(z) \rangle = \int_{z=-\eta}^{\eta} f(z) g(z) dz$. Moreover, the matrix coefficients $a_{0,n}^{(0,2)}$, $a_{1,n}^{(1)}$, $b_{1,n}$ and $c_{1,n}$ are similarly defined.

The weakly nonlinear CMS

By assuming small wave amplitudes and considering the wave velocity field to be time harmonic with angular frequency $\omega$, the latter is described by means of a complex velocity potential $\varphi(x_1,x_2,z;t) = \Re \{ \varphi(x_1,x_2,z) \exp(-i\omega) \}$. In this case, the variational principle (11) simplifies as follows

$$\rho \int_{x_1} \left[ - \frac{1}{g} (-i \omega + \mathbf{U} \cdot \nabla) \varphi + \frac{\partial \varphi}{\partial z} + \frac{1}{g} (\mathbf{U} \cdot \nabla) (-i \omega + \mathbf{U} \cdot \nabla) \varphi \right] d\varphi = 0,$$

$$+ \left[ \frac{\partial \varphi}{\partial z} + \mathbf{U} \cdot \nabla \varphi \right], \quad (20)$$

where, from now on, $\varphi = \varphi(x_1,x_2) \phi$ denotes the complex wave potential. Furthermore, noting that the term

$A = (-i \omega + \mathbf{U} \cdot \nabla)^2 \varphi + (\mathbf{U} \cdot \nabla)(-i \omega + \mathbf{U} \cdot \nabla) \varphi$,

appearing in the integral on the mean free-surface ($z=0$) can also be written as follows:

$$A = -\omega^2 \varphi - 2i \omega (\mathbf{U} \cdot \nabla) \varphi - i \omega (\mathbf{U} \cdot \nabla) \varphi + \mathbf{U} \cdot \nabla \left[ (\mathbf{U} \cdot \nabla) \varphi \right],$$
the variational principle for small-amplitude, time-harmonic wave motion, Eq. (21) is finally put in the form:

\[
\int dt \int dx_1 dx_2 \left\{ \int_{-h(x)}^{z_0} \left( \nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} \right) dz + \left[ \frac{\partial \phi}{\partial z} + \nabla \phi \nabla h \right]_{z=-h(x)} - \left[ A + \frac{\partial \phi}{\partial z} \right]_{z=0} \partial \phi \right\} = 0.
\]  

(22)

Also, in this case the local-mode series expansion of the wave potential is simplified as follows

\[
\phi(x,z) = \sum_{n=1}^{\infty} \phi_n(x) Z_n(z;x),
\]  

(23)

where the functions \( Z_n(z;x), \ n = 0, 1, 2, \ldots \), appearing in Eq. (23), are obtained as the eigenfunctions of local vertical Sturm-Liouville problems (13), formulated with respect to the local depth and the local intrinsic frequency \( \sigma = \omega - U \cdot \mathbf{k} \) in the vertical interval \( -h(x) \leq z \leq h \). Thus, in the examined case the local vertical eigenfunctions are given by

\[
Z_n = \frac{\cosh[k_n(z+h)]}{\cosh(k_n h)}, \quad Z_n = \frac{\cos[k_n(z+h)]}{\cos(k_n h)}, \quad n = 1, 2, \ldots,
\]  

(24)

where the wavenumbers \( k_n \) are obtained as a solution to the local dispersion relation associated with the intrinsic frequency:

\[
\sigma^2 = k_n^2 g \tanh(k_n h) = -k_n g \tan(k_n h).
\]  

(25)

As concerns the sloping-bottom mode \( \phi_{-1} Z_{-1} \), a specific alternative form of the function \( Z_{-1}(z;x) \) used in this case is defined by

\[
Z_{-1} = h \left( \frac{z}{h} - \frac{1}{4} \right),
\]  

(26)

having the following properties:

\[
Z_{-1}'(z=-h) = 1, Z_{-1}'(z=-h) = 0,
\]

and \( Z_{-1}'(z=0) = Z_{-1}(z=0) = 0 \) (see also the discussion by Athanassoulis & Belibassakis 1999). Using the representation (23), in conjunction with the properties of \( Z_n \), we obtain

\[
\frac{\partial \phi}{\partial z} = \mu \phi = \frac{\sigma^2}{g} \phi, \quad \text{on} \quad z = 0.
\]  

(27)

Introducing the above result in the last form of the variational principle, Eq. (22), we obtain

\[
\int dt \int dx_1 dx_2 \left\{ \int_{-h(x)}^{z_0} \left( \nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} \right) dz + \left[ \frac{\partial \phi}{\partial z} + \nabla \phi \nabla h \right]_{z=-h(x)} - \left[ A + \frac{\partial \phi}{\partial z} \right]_{z=0} \partial \phi \right\} = 0.
\]  

(28)

Using the local-mode representation (21) in the variational equation (28), we finally obtain the weakly nonlinear coupled-mode system (CMS) of equations, in the form

\[
\sum_n \left\{ a_m \nabla^2 \phi_n + \left[ b_m + \frac{2i\omega}{g} \right] \nabla \phi + \left[ c_m + \frac{\sigma^2 - \frac{\sigma^2}{g} - \frac{i\omega}{g} \nabla \cdot \mathbf{U} \right] \phi_n + \frac{1}{g} \nabla \cdot \left( \mathbf{U} \left[ \nabla \cdot \mathbf{U} \right] \phi_n \right) = 0, \quad m = -1, 0, 1, \ldots,
\]  

(29)

In Eqs. (29) the coefficients \( a_m, b_m, c_m \) are defined by:

\[ a_m = \langle Z_m, Z_n \rangle, \quad b_m = 2i\omega \nabla Z_n + \nabla Z_n (-h) Z_n^2 \mathbf{U}, \quad c_m = \langle Z_c, Z_n \rangle + \frac{i\omega}{g} \nabla Z_n + \nabla \mathbf{U}, \quad Z_n (-h),
\]  

(30a,b)

The one-equation model

A significant simplification of the above CMS (29) is obtained by keeping only the propagating mode \( n = 0 \) in the local-mode series expansion of the wave potential (23), which essentially describes the propagation features. In this case, the above coupled-mode system is reduced to the following one-equation model on the horizontal plane,

\[
a_{00} \nabla^2 \phi_{00} + \left[ b_{00} + \frac{2i\omega}{g} \right] \nabla \phi_{00} = -\nabla \cdot \left( \mathbf{U} \left[ \nabla \cdot \mathbf{U} \right] \phi_{00} \right) + \left[ c_{00} + \frac{\sigma^2 - \frac{\sigma^2}{g} - \frac{i\omega}{g} \nabla \cdot \mathbf{U} \right] \phi_{00} = 0.
\]  

(31)

The coefficients \( a_{00}, b_{00}, c_{00} \) are given by Eqs. (30) for \( m=0 \), and after processing they become as follows:

\[ a_{00} = \int \frac{Z_0^2 dz}{2k_0} = \frac{1}{2k_0} \tanh(k_0 h) \left( 1 + \frac{2k_0 h}{\sinh(2k_0 h)} \right) = \frac{1}{g} CC_g,
\]  

(32a)

\[ b_{00} = \nabla a_{00} = \frac{1}{g} \nabla CC_g = \frac{1}{g} \left( \nabla Z_0, Z_0 \right) + Z_0 \nabla \mathbf{U} \mathbf{Z} (-h) = k_0 a_{00} + c_{00}.
\]  

(32b)

The coefficient \( c_{00} \) contains terms proportional to first and second horizontal derivatives of the depth function (proportional to bottom slope and curvature), as well as first and second horizontal derivatives of the horizontal current velocity components \( U_x \) and \( U_z \). Using the above expressions of the coefficients in the one-equation model (31) and multiplying by \( g \), it is easily seen that the latter takes the form

\[
\nabla \left( CC_g \nabla \phi_{00} \right) - \nabla \cdot \left( \mathbf{U} \left[ \nabla \cdot \mathbf{U} \right] \phi_{00} \right) + 2i\omega \mathbf{U} \nabla \phi_{00} + \left[ k_0^2 CC_g + g c_{00}^2 + \sigma^2 - \frac{\sigma^2}{g} + i\omega \nabla \cdot \mathbf{U} \right] \phi_{00} = 0,
\]  

(33)

which will be called the modified mild-slope equation for wave scattering by ambient current in general bottom topography. We note here that the model (33), in the case of no current \( U=0 \), exactly reduces to the modified mild slope equation derived by Massel (1993) and Chamberlain & Porter (1995). Furthermore, it is worth noticing here that if the term \( c_{00} \) is omitted, then Eq. (33) reduces exactly to the mild slope equation derived by Kirby (1984), in its time-harmonic form; see Chen et al (2005).
5. NUMERICAL RESULTS AND DISCUSSION

In this section, numerical results are presented for two representative test cases corresponding to a rip current over a sloping beach in shallow water, and to a ring current in constant depth.

The case of rip current

Wave-induced rip currents, created by longshore currents converging into periodic rips and forming independent coastal circulation cells, play an important role in coastal morphodynamics. Also, rip currents, in conjunction with local amplification of wave energy, are responsible for many accidents in beaches. In order to illustrate the effects of a rip current, in conjunction with slow changes of the bathymetry, on the wave scattering, we examine a sloping beach of uniform slope 1/50. Using the similar expressions introduced by Chen et al. (2005), the structure of the rip current $U = (U_x, U_z)$ considered in this example is modelled as follows:

$$U_x = -0.0681 \times F(x/76.2) F(x/76.2), \quad (34a)$$

$$U_z = -0.256 \left[ 2 - (x/76.2)^2 \right] F(x/76.2) \int_{-x/76.2} F(\tau) d\tau, \quad (34b)$$

where $F(\tau) = (2\pi)^{-1/2} \exp(-\tau^2/2)$.

The structure of the current is shown in Figs. 2 and 3, and, its maximum value is selected to be 0.5m/s. We consider unit-amplitude harmonic waves of period $T=14s$ (corresponding to a swell) propagating along the $x_1$-axis (normally to the bottom contours). In this case, the phase velocity of the incident wave is $C=5.5m/s$. The scattered wave field by the above configuration as calculated by means of the solution of the present modified mild-slope equation (using only one mode) is plotted in Fig.2a (phase) and Fig.3a (amplitude). The same result calculated using three modes ($\mu=-1,0,1$) is comparatively plotted in Fig.2b (phase) and Fig.3b (amplitude). In the examined case, a 350m by 200m horizontal domain is considered, discretised by using 151 equidistant points along each horizontal direction. In this case, a significant focusing of wave energy as the shallow-end of the variable bathymetry region is approached is observed, taking place immediately shorewards the formation of the rip current. Also, in this area (where energy is maximized) the greatest differences between the one and three mode solution are observed.

To illustrate the role and the relative significance of the various modes, the amplitudes of the propagating mode $\varphi_0(x)\varphi_y$, the first evanescent mode $\varphi_1(x)\varphi_y$ and the sloping-bottom mode $\varphi_{z-1}(x)\varphi_y$, are plotted in Fig.4 a), b) and c), respectively. We observe in these figures that the main effects of wave refraction and diffraction are carried out by the propagating mode. This fact justifies the use of the simplified, one-equation MMS model to obtain approximate results. However, as shown by
Fig. 4. Modal amplitudes of the scattered wave field by a rip current in a sloping beach region: (a) the propagating mode \((n=0)\), (b) the first evanescent mode \((n=1)\) and (c) the sloping bottom mode \((n=-1)\).

Athanassoulis et al (2003), the applicability of the latter model is restricted to slowly varying bottom topographies. Moreover, in Fig. 4 we observe that the first evanescent mode \(\varphi_1(x,y)\) is one order of magnitude less than the propagating mode. This mode is clearly connected with the non-horizontal part of the bottom. Furthermore, the sloping-bottom mode \(\varphi_{-1}(x,y)\) is of the same order of magnitude, in the swallow area of the bottom.

The case of a vortex-ring current

In this example we consider wave scattering by a vortex ring current in constant depth \(h=10\) m, examined also by Chen et al (2005). Such current structure is commonly seen in open sea and coastal areas, and has important impact on physical and biological processes; see Mapp et al (1985). Following Yoon & Liu (1989), the background current flow associated with the vortex ring is defined by,

\[
U_r = 0, \quad U_\theta = \begin{cases} 
  C_s (r/R_1)^N, & r \leq R_1 \\
  C_s \exp \left[ -1 \frac{(R_2-r)^2}{R_2^2} \right], & r \geq R_1 
\end{cases}
\]

Fig. 5. Real part of the scattered wave field on the free surface: (a) 1 mode solution and (b) 3 modes solution.

where \(U_r\) and \(U_\theta\) denote the radial and tangential components of the horizontal flow \(U\), in a cylindrical-polar coordinate system \((R = |x|, \theta = \tan^{-1}(x_2/x_1))\) with origin at the center of the vortex ring. The following values of the parameters (see Mapp et al 1985) have been selected to describe the vortex-ring: \(C_s = 0.9\) m/sec, \(C_\theta = 1.0\) m/sec, \(N = 2\),
\[
R_1 = 343.706\ m, \quad R_2 = 384.881\ m, \quad R_3 = 126.830\ m.
\]

The vortex ring creates a shearing current with maximum tangential velocity \(1\) m/s, see Fig. 5. We consider unit-amplitude harmonic waves of period \(T=20\) s, propagating along the \(x_1\)-
axis, with phase velocity $C=9.74\text{m/s}$, and scattered by the above vortex-ring current. In the examined case, a 2km by 2km horizontal domain is considered, which is discretised by using 201 equidistant points along each horizontal direction. Numerical results obtained by the present model using one and three modes are shown in Fig.5 as concerns the real part of the wave field on the free surface. The focusing and defocusing of wave energy in the area downwave the vortex ring are well reproduced. In this case, the differences between the two solutions are very small, due to the fact that the bottom is flat, and thus, the slopping bottom mode is zero (by definition). This fact is clearly illustrated in Fig.6, where the amplitude of the propagating (Fig.6a) and the first evanescent mode (Fig.6b) are comparatively plotted. It is seen in this example that the first evanescent mode is one order of magnitude less than the propagating mode and thus, the one-mode solution obtained using the one-equation model Eq. (31) is an excellent approximation.

6. CONCLUDING REMARKS

A coupled-mode model for the wave-current-seabed interaction problem is presented, with application to wave scattering by ambient currents in variable bathymetry regions. Based on an appropriate variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave potential in a finite subregion containing the current variation and the bottom irregularity, the present system can be considered as a generalization of the one derived by Athanassoulis & Belibassakis (2002) for the propagation of waves in variable bathymetry regions. The key feature of the present method is the introduction of an additional mode, describing the influence of the bottom slope, and accelerating the convergence of the local-mode series. If only the propagating mode is retained in the vertical expansion of the wave potential, the above coupled-mode system is reduced to the one-equation enhanced model called the modified mild-slope equation for wave-current flow, generalizing the corresponding mild-slope equation derived by Kirby (1984). Finally, the analytical structure of the present system facilitates its extension to various directions as, e.g., to non-linear wave-current scattering problems and more general current profiles.

REFERENCES


