Keywords: non-uniform waveguides, coupled modes, local-mode series

Abstract. We consider the problem of time-harmonic wave propagation and scattering in a nonuniform waveguide governed by a general second-order, strongly elliptic equation, with variable coefficients. An improved coupled-mode method is presented, based on an enhanced local-mode series for the representation of the wave field, which includes an additional mode accounting for the effects of the boundary slope. The additional mode provides an implicit summation of the slowly convergent part of the local-mode series, rendering the remaining part to be fast convergent. Using the enhanced representation, in conjunction with an appropriate variational principle, a new system of coupled-mode equations is derived for the determination of the unknown modal-amplitude functions. Numerical applications are presented illustrating the role and significance of the additional mode and the efficiency of the present coupled-mode theory, which can be naturally extended to treat propagation and scattering problems in three-dimensional multi-layered waveguides.

1 INTRODUCTION

Wave propagation and scattering in an inhomogeneous waveguide is an interesting mathematical problem finding important applications, as, e.g., underwater acoustic propagation and scattering in shallow water and seismoacoustics [1,2], atmospheric acoustics [3] and others. Similar problems governed by the Helmholtz equation are also encountered in variable cross-section electromagnetic waveguides [4]. Several methods for treating this, generally non-separable, boundary value problem have been proposed, ranging from fully numerical, finite element and finite difference methods, to semi-analytical ones, like wavenumber integration, boundary integral equation and coupled-mode techniques, as well as various asymptotic models, like ray theory and the adiabatic and parabolic approximations; see e.g., [1,2,5]. Fully numerical methods, are computationally intensive and thus, their use is more appropriate for short-range/low-frequency propagation and local scattering problems. These methods have been proved very useful, in the case of general wave propagation problems, for providing benchmark solutions. Boundary element methods are suitable in the case of a homogeneous medium, but they become less practical in the presence of inhomogeneities of the acoustic parameters, due to the difficulties associated with the calculation of the Green's function.

In the present work, we consider the problem of time-harmonic wave propagation and scattering in a nonuniform waveguide (see Fig.1) governed by a general second-order, strongly elliptic equation, with variable coefficients, of the form

\[
\mathcal{L} u = \partial^2_{x_i} \left( a_{ij} \partial_{x_j} u \right) + \nu u = 0 ,
\]  (1)

where \( a_{ij} \) and \( \nu \) are smooth real functions of the spatial variables \( x_i \), and \( a_{ij} > 0 \) is symmetric satisfying condition of uniform ellipticity in the domain \( D \); see, e.g., Ladyzhenskaya [6]. Summation with respect to repeated indices is assumed in Eq. (1). For simplicity, in the present work we consider a two-dimensional \((x_1, x_2)\) waveguide, however the whole analysis is easily extended to 3D. The waveguide is limited above by a general boundary, where homogeneous Dirichlet condition is satisfied,
and below also by a general boundary, where homogeneous, generalised Neumann condition is satisfied, as follows

\[ \partial u / \partial n_x + \sigma u = 0, \quad \text{on} \quad x_2 = h(x_1), \quad \text{where} \quad \partial u / \partial n_x = a_n n_i \left( \partial u / \partial x_i \right), \]

and \( n_i \) denotes the outer normal on \( \partial D \). Moreover, we assume that the 2D waveguide becomes planar in the regions of incidence \((x_1 < a)\) and transmission \((x_1 > b)\), where the functions \( h_1 \) and \( h_2 \) become constant. Thus, the corresponding cross dimension of the waveguide becomes constant and equal to \( H^{(1)} \) and \( H^{(3)} \) (not necessarily equal), respectively. Also, the coefficients in the above Eqs. (1,2,3) are assumed to be constant in these regions, but they may have different values. The waveguide is excited by incident plane waves propagating in the positive \( x_1 \) direction, as schematically shown in Fig.1.

In the present work an improved coupled-mode method is presented, based on an enhanced local-mode series for the representation of the wave field, which includes an additional mode accounting for the effects of the slope of the lower boundary where Neumann condition is satisfied. The additional mode provides an implicit summation of the slowly convergent part of the local-mode series, rendering the remaining part to converge much faster, like \( O(n^{-4}) \), where \( n \) is the mode order. Using the enhanced representation, in conjunction with an appropriate variational principle, a system coupled-mode equations is derived for the determination of the unknown modal-amplitude functions. Numerical applications are presented illustrating the role and significance of the additional mode and the efficiency of the present coupled-mode theory, which can be naturally extended to treat propagation and scattering problems in three-dimensional multi-layered waveguides.

\section*{2 FORMULATION OF THE PROBLEM}

On the basis of the above assumptions the domain \( D \) is decomposed to three parts \( D^{(m)}, \ m = 1, 2, 3 \) (Fig. 1), as follows: \( D^{(1)} \) is the subdomain characterized by \( x_1 < a \) where the cross dimension is constant and equal to \( H^{(1)} \), \( D^{(3)} \) is the subdomain characterized by \( x_1 > b \) \((b > a)\) where the cross dimension is constant and equal to \( H^{(3)} \), and \( D^{(2)} \) is the variable cross section subdomain lying between \( D^{(1)} \) and \( D^{(3)} \). A similar decomposition is also applied to the (upper and lower) boundaries. Complete normal-mode type representations of the wave field in the regions of incidence and transmission can be derived by separation of variables, as follows,

\[ u^{(1)}(x_1,x_2) = \sum_{n=1}^{\infty} \left( A_n^{(1)} \exp(ik_n^{(1)}x_1) + B_n^{(3)} \exp(-ik_n^{(1)}x_1) \right) Z_n^{(1)}(x_2), \quad \text{for} \quad x_1 < a, \]

\[ u^{(2)}(x_1,x_2) = \sum_{n=1}^{\infty} \left( A_n^{(2)} \exp(ik_n^{(2)}x_1) + B_n^{(3)} \exp(-ik_n^{(2)}x_1) \right) Z_n^{(2)}(x_2), \quad \text{for} \quad x_1 > b. \]
The functions \( Z_n^{(m)}(x_2) \), \( m = 1, 3 \), in the above equations are obtained as the eigenfunctions of the following, regular Sturm-Liouville problems

\[
\frac{d^2 Z_n^{(m)}(x_2)}{dx_2^2} + \lambda_n^{(m)} Z_n^{(m)} = 0 \quad , \quad h_l^{(m)} \leq x_2 \leq h_r^{(m)} , \quad m = 1, 3 \tag{5a}
\]

\[
Z_n^{(m)}(x_2 = h_l^{(m)}) = 0 , \quad m = 1, 3 \tag{5b}
\]

\[
-\frac{d^2 Z_n^{(m)}(x_2 = h_l^{(m)})}{dx_2^2} + \sigma^{(m)} Z_n^{(m)}(x_2 = h_l^{(m)}) = 0 , \quad m = 1, 3 \tag{5c}
\]

where \( a_{12}^{(m)} > 0 \). The solution of Eqs.(5) generates orthonormal vertical bases in \( D^{(m)}, m = 1, 3 \), respectively; see, e.g., Coddington & Levinson [7], Titchmarsh [8]. The numbers \( \{k_n^{(m)}, n = 1, 2, 3, ...\} , m = 1, 3 \), appearing in the expansions (4) are obtained through the corresponding eigenvalues \( \lambda_n^{(m)} \), respectively, as follows

\[
k_n^{(m)} = \sqrt{\nu^{(n)} - \lambda_n^{(m)}} / a_{12}^{(m)} , \quad n = 1, 2, 3, ..., \quad m = 1, 3 \tag{6}
\]

Clearly, the eigenfunctions \( Z_n^{(m)}(x_2) \) of the above problems are real functions, defined as

\[
Z_n^{(m)}(x_2) = \sin \left( \sqrt{\lambda_n^{(m)} / a_{12}^{(m)}} (x_2 - h_l^{(m)}) \right) , \quad m = 1, 3 \tag{7a}
\]

and the eigenvalues \( \{\lambda_n^{(m)}, n = 1, 2, 3, ...\} , m = 1, 3 \), are obtained as the roots of the following equations

\[
\sqrt{\lambda_n^{(m)} a_{12}^{(m)}} \cos \left( \sqrt{\lambda_n^{(m)} / a_{12}^{(m)}} (h_l^{(m)} - h_r^{(m)}) \right) - \sigma^{(m)} \sin \left( \sqrt{\lambda_n^{(m)} / a_{12}^{(m)}} (h_l^{(m)} - h_r^{(m)}) \right) = 0 , \quad m = 1, 3 \tag{7b}
\]

From the properties of regular Sturm-Liouville problems [7], the eigenvalues \( \{\lambda_n^{(m)}, n = 1, 2, ...\} \) are discrete, infinite, with continuously increasing moduli, and thus, the corresponding parameters \( \{k_n^{(m)}, n = 1, 2, 3, ...\} \), as obtained from Eq. (6), are subdivided into a finite real subset \( \{k_n^{(m)}, n = 1, 2, 3, ...N_p^{(m)}\} \) and an infinite imaginary one \( \{ik_{n}^{(m)}, n = N_p^{(m)} + 1, ..., \} \), where \( N_p^{(m)}, m = 1, 3 \), denotes the number of propagating modes in \( D^{(m)} \).

Clearly, in order for the wave field to remain bounded at infinity, the coefficients of the expansion \( A_{11}^{(i)} = 0, n > N_p^{(i)} \) and \( B_{12}^{(i)} = 0, n > N_p^{(i)} \). The terms \( \exp \left( -ik_{n}^{(i)} x_2 \right) Z_n^{(i)}(x_2) , n > N_p^{(i)} \), and \( \exp \left( ik_{n}^{(i)} x_2 \right) Z_n^{(i)}(x_2) , n > N_p^{(i)} \), are the evanescent modes in \( D^{(i)}, m = 1, 3 \), respectively, and decay exponentially at large distances in these two semi-infinite strips.

We note here that in the case of Helmholtz equation in a homogeneous waveguide, the coefficients \( a_{12} = a_{23} = 1 \), \( a_{23} = a_{23} = 0 \) and \( \nu = k^2 = (\omega / c)^2 \), where \( \omega \) denotes the angular wave frequency and \( c \) the constant propagation speed. In the case of a hard lower boundary (\( \sigma = 0 \)) the above expressions reduce to the following well known results (see, e.g., [2])

\[
\sqrt{\lambda_n^{(m)}} = (n - 0.5) \pi / (h_r^{(m)} - h_l^{(m)}) , \quad Z_n^{(m)}(x_2) = \sin \left( \sqrt{\lambda_n^{(m)}} (x_2 - h_l^{(m)}) \right) , \quad n = 1, 2, 3, ..., \quad m = 1, 3 \tag{8}
\]

and
\[ k_n^{(m)} = \sqrt{k^2 - \left( (n - 0.5)\pi / H^{(m)} \right)^2}, \quad n = 1, 2, 3, \ldots , \quad \text{where} \quad H^{(m)} = H_u^{(m)} - H_r^{(m)}, \quad m = 1, 3, \quad . \] 

The present waveguide is excited by plane incident waves propagating in the positive \( x_1 \)-direction. Thus, the wave field in subregion \( D^{(i)} \) consists of incident and reflected waves and in subregion \( D^{(o)} \) by outgoing waves only, i.e. \( B_u^{(i)} = 0 \). Also, the coefficients \( A_u^{(i)} \) of the incident wave field are considered to be known (given). Consequently, the unknowns in the two semi-infinite strips of the studied problem are \( \{ B_u^{(i)}, n = 1, 2, \ldots \} \) determining the reflected wave field and \( \{ A_u^{(o)}, n = 1, 2, 3, \ldots \} \) determining the transmitted wave field.

By exploiting the representations (2) and (3) the problem can be formulated as a transmission boundary value problem in the bounded subdomain \( D^{(3)} \), satisfying Eq. (1), (2) and (3), and the following matching conditions:

\[
\begin{align*}
\Delta u^{(2)}(x_1, x_2) &= \Delta u^{(1)}(x_1, x_2), \quad \frac{\partial u^{(2)}}{\partial N_L^{(2)}} = -\frac{\partial u^{(1)}}{\partial N_L^{(1)}}, \quad x_1 = a, \quad , \quad h_1^{(i)} \leq x_2 \leq h_1^{(o)}, \quad (10) \\
\Delta u^{(2)}(x_1, x_2) &= \Delta u^{(1)}(x_1, x_2), \quad \frac{\partial u^{(2)}}{\partial N_L^{(2)}} = -\frac{\partial u^{(1)}}{\partial N_L^{(1)}}, \quad x_1 = b, \quad , \quad h_1^{(i)} \leq x_2 \leq h_1^{(o)}, \quad (11)
\end{align*}
\]

where \( \partial u^{(m)}/\partial N_L^{(m)} \) denote generalized normal derivatives (as defined by Eq. 3c) on the boundary \( \partial D^{(m)}, m = 1, 2, 3, \) of each subdomain, and the normal vectors are always directed outwards. Due to the assumed smoothness of the coefficients \( a_{ij} \) the above continuity conditions (10b) and (11b) reduce to continuity of classical horizontal derivatives of the wave field on the vertical interfaces, i.e.

\[
\frac{\partial u^{(2)}}{\partial x_i} = \frac{\partial u^{(1)}}{\partial x_i}, \quad x_1 = a, \quad h_1^{(i)} \leq x_2 \leq h_1^{(o)}, \quad \text{and} \quad \frac{\partial u^{(2)}}{\partial x_i} = \frac{\partial u^{(1)}}{\partial x_i}, \quad x_1 = b, \quad h_1^{(i)} \leq x_2 \leq h_1^{(o)}, \quad (12)
\]

### 3 VARIATIONAL FORMULATION OF THE PROBLEM

We proceed to formulate a functional \( \mathcal{F} \) allowing us to state a variational formulation of the transmission problem. The admissible function space for the wave field \( u^{(3)}(x_1, x_2) \in D^{(3)} \), denoted simply from now on as \( u(x_1, x_2) \), is defined to be:

\[
\mathcal{A}(D^{(3)}) = \left\{ u \in \left[ C^2(D^{(3)}) \cap C^1(D^{(3)}) \right] : u(x_1, x_2 = h_3) = 0 \right\}.
\]

We consider the following functional

\[
\mathcal{F}(u^{(2)}, \{ B_u^{(1)} \}, \{ A_u^{(1)} \}) = \frac{1}{2} \int_{D^{(3)}} \left[ a_{ij} \left( \frac{\partial u^{(2)}}{\partial x_i} \right) \left( \frac{\partial u^{(2)}}{\partial x_j} \right) - \nu u^{(2)} \right] dV + \frac{1}{2} \int_{\partial D^{(3)}} \sigma \left( u^{(2)} \right) dS + \\
+ \int_{\partial D^{(3)}} \left( u^{(2)} - \frac{1}{2} u^{(1)} \right) \frac{\partial u^{(2)}}{\partial N^{(1)}} dS + \int_{\partial D^{(3)}} \left( u^{(2)} - \frac{1}{2} u^{(1)} \right) \frac{\partial u^{(2)}}{\partial N^{(1)}} dS + f, \quad (14)
\]

where \( \partial D^{(m)}, \partial D^{(m)} \) denote the upper and lower boundaries of the subdomain \( D^{(m)}, m = 1, 2, 3, \) and \( \partial D^{(2)}, \partial D^{(2)} \) denote the vertical interfaces at \( x_1 = a \) and \( x_1 = b \), separating the three subdomains. Also, \( f \) in Eq. (14) is an appropriate constant term. The present transmission problem admits of an equivalent variational formulation expressed by

\[
\delta \mathcal{F}(u^{(2)}, \{ B_u^{(1)} \}, \{ A_u^{(1)} \}) = 0. \quad (15)
\]
Using Green’s theorem, in conjunction with Eq. (13), the above equation takes the form

\[ -\int_{\partial D} \left( \frac{\partial}{\partial x} \left( \alpha \frac{\partial u}{\partial x} \right) + v u \right) + \int_{V} \delta u^{(2)} dV = \int_{\partial D} \left( \frac{\partial u^{(2)}}{\partial N^{(2)}} + \sigma u^{(2)} \right) dS + \int_{\partial D} \left( \frac{\partial u^{(2)}}{\partial N^{(1)}} \right) dS + \int_{\partial D} \left( u^{(2)} - u^{(1)} \right) \delta \frac{\partial u^{(1)}}{\partial N^{(1)}} dS + \int_{\partial D} \left( u^{(2)} - u^{(1)} \right) \delta \frac{\partial u^{(1)}}{\partial N^{(1)}} dS + \int_{\partial D} \left( u^{(2)} - u^{(1)} \right) \delta \frac{\partial u^{(1)}}{\partial N^{(1)}} dS + \int_{\partial D} \left( u^{(2)} - u^{(1)} \right) \delta \frac{\partial u^{(1)}}{\partial N^{(1)}} dS = 0. \] (16)

The usefulness of the above variational principle hinges on the fact that it leaves us the freedom to choose any particular representation for the unknown field \( u(x_1, x_2) \in D^{(2)} \). In this way, a variety of possible algorithms for the numerical solution of the present wave problem can be constructed. One possible choice, facilitating the treatment of studied wave problems in general domains, without restrictions as regards the bottom slope or curvature, will be presented in the following section.

3 AN ENHANCED LOCAL MODE REPRESENTATION OF THE WAVE FIELD

A standard spectral-type representation based on local-modes (briefly denoted by SR) of the wave field \( u(x_1, x_2) \in D^{(2)} \) is defined by

\[ u(x_1, x_2) = \sum_{n=1}^{\infty} U_n(x_1) Z_n(x_2, x_1) \] (17)

The family of local vertical basis functions \( \{ Z_n(x_2, x_1) \}, n = 1, 2, 3, \ldots \) appearing in the above expansion, which are parametrically dependent on \( x_1 \), is obtained by formulating and solving local, vertical Sturm-Liouville problems in the \( x_2 \)-intervals \([h_i(x_1), h_j(x_1)]\), for each horizontal position \( (x_1 = \text{const}) \), as follows

\[ \frac{d^2}{dx_2^2} \left( a_{22} \frac{dZ_n}{dx_2} \right) + \lambda_n Z_n = 0, \quad h_i(x_1) \leq x_2 \leq h_j(x_1), \] (18a)

\[ Z_n(x_2 = h_i(x_1)) = 0, \] (18b)

\[ -\tilde{a}_{22}(x_1) \frac{dZ_n(x_2 = h_j(x_1))}{dx_2} + \sigma(x_1) Z_n(x_2 = h_j) = 0, \] (18c)

where \( \tilde{a}_{22}(x_1) = a_{22}(x_1, x_2 = h_i(x_1)) \). Thus, the standard local-mode representation of the wave field in the middle domain \( D^{(2)} \), satisfies the Dirichlet boundary condition (treated as an essential condition) and serves as a basis of the admissible function space \( \mathcal{A}(D^{(2)}) \).

However, any finite truncation of the series (17) is incompatible with the sloping boundary condition (1c), whenever \( dh_i(x_1)/dx_1 \neq 0 \), rendering the above series to converge only in an \( L_2 \)-sense, and the coefficients \( U_n \) to decay like \( O(n^{-2}) \). To remedy this inconsistency, an additional mode is introduced, denoted by \( U_0(x_1) Z_0(x_2, x_1) \) which is called the sloping-boundary mode. Thus, the enhanced local-mode series (briefly denoted by ER) becomes,

\[ u(x_1, x_2) = U_0(x_1) Z_0(x_2, x_1) + \sum_{n=1}^{\infty} U_n(x_1) Z_n(x_2, x_1) = \sum_{n=0}^{\infty} U_n(x_1) Z_n(x_2, x_1) \] (19)

The vertical structure of the sloping-boundary mode \( Z_0(z, x) \), is any sufficiently smooth function defined in the local vertical intervals \([h_i(x_1), h_j(x_1)]\), satisfying the following conditions for each \( x \in (a, b) \)
Specific convenient forms of the function \( Z_0(z;x) \) are given by low-degree polynomials, as

\[
Z_0(x_2 = h(x_1)) = 0, \quad \partial Z_0(x_2 = h(x_1))/\partial x_2 = 0, \quad Z_0(x_2 = h(x_1)) = 0, \quad \frac{dZ_0}{dx_2}(x_2 = -h(x_1)) = 1. \tag{20}
\]

Thus, \( U_t(x_1)Z_0(x_2;x_1) \) is the additional degree of freedom in the variable cross-section subdomain \( D^{(3)} \) permitting the consistent satisfaction of the boundary condition (3). The amplitude of the additional mode is given by

\[
U_t(x_1) = \frac{du(x_2 = h(x_1))}{dx_2}, \tag{22}
\]

and consequently, it becomes zero in the case of a flat, Neumann lower boundary, i.e. when \( dh_1/\partial x_1 = 0 \) and \( \sigma = 0 \). The effect of the additional mode is to significantly increase the rate of decay of \( Z_n - \) Fourier coefficients of the wave potential (modal amplitudes). In this case, the later modes exhibit a rapid decay rate: \( |U_n(x_1)| \leq C(x_1) n^{-4}, \ n \to \infty \), for each \( x_1 \in [a, b] \). The bound \( C(x_1) \) is a continuous function on \([a, b] \) and, thus, the previous estimate is global: \( |U_n| = O(n^{-4}) \). The effect of the additional mode is to significantly increase the rate of decay of the standard series is only \( |U_n| = O(n^{-2}) \). The proof of the above is based on repetitive use of integration by parts, in conjunction with the properties of the Sturm-Liouville system, Eqs. (18).

Numerical results illustrating the above fast convergent properties of the enhanced local-mode series in various wave propagation and scattering problems have been presented by the authors. Details concerning the wave-propagation problem in variable bathymetry can be found in Ref. [9], [10], and concerning the underwater acoustic problems in [11], [12]. An important consequence of the fast convergence of the local-mode series is that it permits the use of the present consistent coupled-mode model, after appropriate extensions, to consistently and efficiently treat nonlinear and/or more complicated problems, as e.g. nonlinear water waves in variable bathymetry regions [13], [14], hydroelastic applications involving large floating bodies [15],[16], and interaction and problems involving rotational vector wave fields, as, e.g., propagation and scattering problems in curved electromagnetic waveguides [17].

5 THE COUPLED-MODE SYSTEM

Substituting the enhanced local mode representation (19) in the variational principle (16), and expressing the variation of the unknown field \( u(x_1,x_2) \in D^{(3)} \) through the variations of the modal amplitudes

\[
\delta u(x_1,x_2) = \sum_{n=0}^{\infty} Z_n(x_2;x_1) \delta U_n(x_1)
\]

the third term in the left hand side of Eq. (16) is dropped (as a consequence of the fact that \( \delta u = 0 \) for points on \( \partial D^{(3)} \) since \( Z_n(x_2 = h(x_1)) = 0 \). Next, by considering only the variations \( \delta U_n, n = 0,1,2,... \), in \( a < x < b \), we obtain from the first two terms of Eq. (16) the following coupled-mode system (CMS) of second-order ordinary differential equations, with respect to the mode amplitudes, \( U_n(x_1), n = 0,1,2,... \),

\[
\sum_{n=0}^{\infty} a_{mn}(x_1) \frac{d^2 U_n(x_1)}{dx_1^2} + b_{mn}(x_1) \frac{dU_n(x_1)}{dx_1} + c_{mn}(x_1) U_n(x_1) = 0, \quad m = 0,1,2,3... \tag{23}
\]

The \( x_1 \)-dependent coefficients \( a_{mn}, b_{mn}, c_{mn} \) are defined in terms of \( Z_n(x_2;x_1) \) in \( a < x < b \), and are given by

\[
a_{mn}(x_1) = \left\langle a_{n1} Z_n, Z_n \right\rangle, \tag{24a}
\]
\[ b_{mn}(x_i) = \left\{ \frac{\partial a_{11}^n}{\partial x_i} Z_n(x_i) + 2 \left( \frac{\partial Z_n}{\partial x_i} \right) + 2 \left( \frac{\partial a_{12}^n}{\partial x_i} \right) \right\} + \left( \frac{\partial a_{12}^n}{\partial x_i} + \frac{\partial a_{21}^n}{\partial x_i} \right) Z_n(x_i), \quad (24b) \]

\[ c_{mn}(x_i) = \left\{ a_{12} \frac{\partial^2 Z_n(x_i)}{\partial x_i^2} + 2 a_{12} \frac{\partial Z_n}{\partial x_i} + a_{12} \frac{\partial Z_n}{\partial x_i} + a_{21}^{g,n} \frac{\partial Z_n}{\partial x_i} + \left( \frac{\partial a_{12}^n}{\partial x_i} + \frac{\partial a_{21}^n}{\partial x_i} + \frac{\partial a_{22}^n}{\partial x_i} \right) Z_n(x_i) \right\} \quad (24c) \]

where \( <f,g> = \int_{z=-h}^{z=0} f(z)g(z)dz \) denotes the inner product of \( L_2 \) space of functions in the vertical intervals \( (h_a(x_i), h_b(x_i)) \). The coefficients \( \beta_{mn}, \gamma_{mn} \) of the CMS are associated with the boundary terms and are given by

\[ \beta_{mn} = a_{11} \frac{dh Z_n(-h_i) Z_n(-h_i)}{dx_i} \sqrt{1 + (dh/dx_i)^2}, \quad (25a) \]

\[ \gamma_{mn} = \left( a_{11} \frac{dh Z_n(x_i, x_2 = h_i)}{dx_i} + a_{12} \frac{dh Z_n(x_i, x_2 = h_i)}{dx_2} \right) \frac{Z_n(-h_i)}{\sqrt{1 + (dh/dx_i)^2}} - \sigma(x_i) Z_n(x_i, x_2 = h_i). \quad (25b) \]

The CMS (23) contains an additional equation (\( m=0 \)), associated with the sloping-boundary mode, and produces solutions consistent with the boundary conditions. Furthermore, from the last four terms in the left-hand side of the variational equation (16), defined on the vertical interfaces \( \partial D^{(2)}, \partial D^{(3)} \), we obtain the following end-conditions for the mode amplitudes \( u_n(x_i, x_2) \), at \( x_i = a \) and \( x_i = b \),

\[ C^{(m)}(dU_n / dx_i) + D^{(m)}u_n = F^{(m)} \quad n = 0,1,2,... \quad n = 0,1,2,3... \quad m = 1,3 \]

where the coefficients \( C^{(m)}, D^{(m)}, F^{(m)}, m = 1,3 \), are defined in terms of the physical parameters at the endpoints \( x_i = a, x_i = b \), respectively.

### 6 NUMERICAL RESULTS AND DISCUSSION

In this section, the discrete scheme for the numerical solution of the CMS, Eq. (23), is introduced and numerical results are presented. By truncating the ER local-mode series, keeping a finite number of terms (modes), retaining the sloping-boundary mode, the propagating modes and a number of first evanescent modes, the wave field is approximated as follows

\[ u(x_i, x_2) = \sum_{n=0}^{N} U_n(x_i) Z_n(x_2; x_i). \quad (27) \]

The discrete system is obtained by using central, second-order finite differences to approximate the derivatives of the unknown modal-amplitude functions \( U_n(x_i), n = 0,1,2,... \), appearing in the CMS, for \( m=0,1,...,N \). Discrete boundary conditions are obtained from Eqs. (26) using second-order, forward and backward differences to approximate \( x_i \)-derivatives at the ends \( x_i = a \) and \( b \) of \( D^{(2)} \). Thus, the discrete scheme is uniformly of second order in the horizontal direction. On the basis of the above considerations, the coupled-mode system of differential equations is finally reduced to a linear algebraic system. The coefficient matrix of the system is block structured, with three-diagonal subblocks, and has a total dimension \( N_y = (N+1) (N_y + 1) \), where \( N_y \) is the number of segments subdividing the \( x_i \)-interval \([a, b] \). The forcing associated with the known incident wave field appears only at the left endpoint \( x_i = a \). All numerical results presented in this section are based on the choice (21) for the function \( Z_n(x_2; x_i) \). However, extensive numerical experimentation using other forms for \( Z_n(x_2; x_i) \) has shown that the final results concerning the wave field, as obtained by the present ER, remain the same. As a simple example, we consider here concerning acoustic propagation and scattering in an underwater environment, governed by the Helmholtz equation \( (a_{11} = a_{12} = 1, \quad a_{21} = a_{11} = 0, \quad \nu = k^2 = (\omega / c)^2 ) \) in a homogeneous waveguide, limited above by a horizontal \( (h_0 = 0) \) acoustically soft boundary and below by an acoustically hard boundary \( (\sigma = 0) \).
Fig. 2. Real part (top) and imaginary (bottom subplot) of wave field excited by the 1st mode, in an ocean-acoustic waveguide, as calculated by the present CMS. Colorbar is used to indicate the wave field magnitude.

Fig. 3. Vertical distributions of the wave field excited by the 1st mode, as calculated by the present CMS, (a) at the entrance, (b) middle, and (c) exit of the waveguide. Real part of the wave field is shown by using solid lines and imaginary part by dashed lines, respectively. The position of the bottom boundary is denoted by thick horizontal lines.

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<th>mode</th>
<th>$\lambda_{1}^{(1)}$ (m)</th>
<th>$\lambda_{1}^{(3)}$ (m)</th>
<th>$B_{1}^{(1)}$</th>
<th>$B_{2}^{(1)}$</th>
<th>$B_{3}^{(1)}$</th>
<th>$A_{1}^{(1)}$</th>
<th>$A_{2}^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>50.1</td>
<td>51.6</td>
<td>0.30</td>
<td>0.87</td>
<td>0.67</td>
<td>0.69</td>
<td>1.55</td>
</tr>
<tr>
<td>2</td>
<td>51.6</td>
<td>75.5</td>
<td>0.87</td>
<td>0.26</td>
<td>0.80</td>
<td>1.46</td>
<td>0.18</td>
</tr>
<tr>
<td>3</td>
<td>55.0</td>
<td>-</td>
<td>0.70</td>
<td>0.81</td>
<td>0.40</td>
<td>0.57</td>
<td>1.27</td>
</tr>
</tbody>
</table>

Table 1: Calculated values of the coefficients of the expansions, Eq. (4), in the two semi-infinite strips, for wave field excited by the 1st, 2nd, and 3rd mode, respectively, and corresponding mode wavelengths.
In this case, the depth function $H(x_i) = -h_i(x_i)$ presents a variation from 150m to 50m over 800m horizontal distance, corresponding to a steep upslope environment in relatively shallow water. The mean bottom slope is 12.5% and the maximum bottom slope, appearing in the middle of the domain, is 60%. This acoustic environment is characterised by a constant sound speed $c=1500\text{m/sec}$ (corresponding to water). An incident wave of $30\text{Hz}$ frequency, is considered. In this example the number of propagating modes in the two semi-infinite strips ($x_i \leq a = 0\text{m}$ and $x_i \geq b = 800\text{m}$) are $N_p^{(1)} = 3$ and $N_p^{(2)} = 2$, respectively. Numerical results concerning the calculation of the wave field are presented in Figs. 2-5 and Table 1. A horizontal grid of $N_x = 251$ equidistant points has been used, corresponding to more than 15 gridpoints per wavelength, and 15 modes have been retained in the local-mode series (27). In particular, in Fig. 2, a contour plot of the wave field excited by the $1^{st}$ incident mode, in the above ocean-acoustic waveguide, is shown. Extension of these lines below the lower boundary is maintained in order to better visualize the fulfillment of the bottom boundary condition, and we clearly observe that contour lines intersect the bottom profile perpendicularly. The corresponding vertical
distributions of the wave field are presented in Fig. 3, at three horizontal positions, at the entrance, the middle, and exit of the waveguide. Further, the modulus of the wave field, which in the case of acoustic propagation is proportional to the energy in the domain, is shown in Fig 4, when the waveguide is excited by the 1st, 2nd and 3rd incident modes (which for this frequency are the only ones propagating in the incidence region). A color bar is used to indicate the wave energy magnitude, and we can observe the increase of energy near the regions where the vertical cross section of the waveguide is reduced. Finally, in Fig.5, the moduli of the modal-amplitude functions \( U_n(x) \), \( n = 0, 1, 2, \ldots \), are plotted vs. mode-number \( n \), as calculated by the present CMS. We clearly observe the significance of the propagating modes, in comparison to the evanescent modes. In the same figure the curve \( \alpha n^{-4} \) is drawn (where \( \alpha \) is a constant), using a thick dashed line. This curve bounds the maxima of the amplitudes of the modal functions as obtained by the present SR, indicating that the rate of decay of the mode amplitudes is \( [U_n] = O(n^{-4}) \). The last result ensures the absolute and uniform convergence of the SR series, and its derivatives, up to (and including) the boundaries, and permits us to obtain accurate solutions in the general case keeping a small number of evanescent modes (5-6) in the truncated enhanced local-mode series expansions.

7 CONCLUSIONS

In this work an improved coupled-mode method is presented for the efficient solution of the problem of time-harmonic wave propagation and scattering in a non-uniform waveguide, governed by a general second-order, strongly elliptic equation, with variable coefficients. Our method is based on an enhanced local-mode series for the representation of the wave field, which includes an additional mode accounting for the effects of the boundary slope. Numerical applications are presented illustrating the role and significance of the additional mode and the efficiency of the present coupled-mode theory, which can be naturally extended to treat wave propagation and scattering problems in three-dimensional, multi-layered waveguides.

REFERENCES