A normal-mode solution for 3D acoustic scattering from a cylindrical island

G.A. Athanassoulis, A.M. Prospathopoulos, K.A. Belibassakis
Department of Naval Architecture and Marine Engineering, National Technical University of Athens, P.O. Box 64070, 15710 Zografos, Greece.

A normal-mode solution is presented for the three-dimensional (3D) problem of acoustic scattering from a non-penetrable cylindrical island in shallow water. The acoustic field is generated by an harmonic point source, located outside the island. The ocean environment around the island is considered range independent, with arbitrary sound speed profile, and the bottom is assumed to be rigid. A new decomposition technique is applied to the representation of the acoustic pressure field, which improves the efficiency of normal-mode calculations, enabling thus the extension of the solution to higher frequencies. Results concerning the transmission loss around an acoustically soft and an acoustically hard cylindrical island are presented for moderate frequencies (500Hz and 1kHz) and isovelocity ocean environment. Generalizations to more realistic ocean environments are discussed.

1. INTRODUCTION

Three-dimensional analytical models of acoustic propagation, providing a full description of the field in all spatial coordinates, are of special importance in computational ocean acoustics. They provide physical insight, giving the exact dependence of the pressure field not only on range and depth but also on azimuth, and may serve as benchmarks for numerical models as well. Of particular interest are the 3D analytical models dealing with sound propagation and scattering around steep seamounts and islands, because of the difficulty due to the presence of a strong horizontal discontinuity of the acoustic parameters locally.

An extensive survey of the literature concerning mainly methods based on the full-wave (Helmholtz) equation for acoustic scattering from an obstacle in the sea, as well as for acoustic propagation in an ocean waveguide with a steep local irregularity of its bottom, can be found in a recent work by Athanassoulis and Prospathopoulos[4]. In this work, as well as in Ref. [3], normal-mode series solutions have been developed for the problem of 3D acoustic scattering of a source-generated field from a non-penetrable cylindrical island in an, otherwise, range independent waveguide, with arbitrary continuous sound speed profile. The geometrical configuration of the studied environment is depicted in Fig. 1.
From the point of view of numerical calculations, the series-expansion solution derived in Ref. 1 has a gradually degraded efficiency as the characteristic wavenumbers increase. However, since the series expansion is an exact analytic solution of the problem, appropriate special treatments permit us to improve the numerical efficiency and widen the range of its applicability. A possible way of doing this is an appropriate decomposition of the pressure field. Such a technique is applied to the present work, which is organized as follows: In Section I the mathematical formulation of the problem is described and the normal-mode expansion derived in Ref. 1 is presented. In Section II a decomposition is applied to the representation of the acoustic pressure field, improving the efficiency of normal-mode calculations. Results concerning the transmission loss around an acoustically soft and an acoustically hard cylindrical island are given in Section III. In the same section, generalizations to more realistic ocean environments are discussed.

2. THE NORMAL-MODE SOLUTION

Let us consider the geometry of the environment illustrated in Fig. 1. A vertical cylinder of radius \( a \) is surrounded by a water layer of constant depth \( h \), which is confined between a rigid (perfectly reflecting) bottom \( \partial D_B \) and a pressure-release sea surface \( \partial D_F \). The cylinder is extended from the bottom up to the sea surface, and its lateral (wetted) surface \( \partial D_I \) can vary from an acoustically soft to an acoustically hard one. The geometric domain occupied by the water mass is denoted by \( D \). Somewhere in \( D \) a point source of unit strength is located, emitting monochromatic sound waves with angular frequency \( \omega \).

A cylindrical-polar coordinate system is introduced having as \( z \)-axis the axis of the cylinder pointed upward, and origin at the intersection of \( z \)-axis and the bottom of the waveguide. The source point is denoted by \( r_s=(r_s, z_s, 0) \), and the generic field point is represented by \( r=(r, z, \theta) \). The sound speed profile in water is assumed to be range independent, that is \( c=c(z) \), and the density \( \rho \) is considered constant.

---

FIG. 1. Geometrical configuration
The field equation describing acoustic wave propagation in the above environment is the inhomogeneous Helmholtz equation

$$\nabla^2 p(r, z, \theta) + k^2 p(r, z, \theta) = -\frac{\delta(r - r_s)\delta(z - z_s)\delta(\theta)}{r} \quad \text{in } D, \quad (1)$$

where $p(r, z, \theta)$ is the acoustic pressure (factoring out the harmonic time dependence $e^{-i \omega t}$), and $k = k(z) = \omega/c(z)$ is the wavenumber.

The acoustic pressure should also satisfy the boundary conditions

$$p = 0 \quad \text{on } \partial D_F \ (z = h), \quad (2a)$$

$$\frac{\partial p}{\partial z} = 0 \quad \text{on } \partial D_B \ (z = 0), \quad (2b)$$

given $p$ or $\frac{\partial p}{\partial r}$ or $\beta p + \gamma \frac{\partial p}{\partial r}$ on $\partial D_I \ (r = a), \quad (2c)$

and an appropriate (Sommerfeld) radiation condition at infinity ($r \rightarrow \infty$), expressing that the pressure field at infinity behaves like a system of cylindrical outgoing waves.

In a previous work\cite{1} the following representation theorem has been proved:

**Theorem:** The field $p(r, z, \theta)$, $(r, z, \theta) \in D$, satisfies the inhomogeneous Helmholtz equation (1), the boundary conditions (2a,b) and a Sommerfeld radiation condition at infinity of the form (9b) if and only if it admits of the representation

$$p(r, z, \theta) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e_m A_{nm} H_n^{(1)}(\lambda_n k_0 r) g_n(z) \cos m\theta$$

$$+ \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e_m H_n^{(1)}(\lambda_n k_0 r >) g_n(z) H_n^{(1)}(\lambda_n k_0 r <) g_n(z) \cos m\theta, \quad (3)$$

where $e_0 = 1/2$ and $e_m = 1$, $m > 0$, $r_s = \max\{r, r_s\}$, $r_c = \min\{r, r_s\}$, $g_n(z)$ are the normalized eigenfunctions of the vertical problem, corresponding to the range independent axisymmetric environment outside the cylinder, and $A_{nm}$ are "arbitrary" constants.

It has also been shown\cite{1} that, in the cases of a soft and a hard cylindrical surface of the island, the pressure field is expressed by the form

$$p(r, \theta, z) = \frac{1}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e_m H_n^{(1)}(\lambda_n k_0 r >) g_n(z) \left[ H_n^{(2)}(\lambda_n k_0 r <) - \Lambda_m(\lambda_n k_0 a) H_n^{(1)}(\lambda_n k_0 r <) \right] g_n(z) \cos m\theta, \quad (4)$$

where

$$\Lambda_m(\lambda_n k_0 a) = \frac{H_n^{(2)}(\lambda_n k_0 a)}{H_n^{(1)}(\lambda_n k_0 a)}, \quad \Lambda_m^{\text{hard}}(\lambda_n k_0 a) = \frac{\Phi_n^{(2)}(\lambda_n k_0 a)}{\Phi_n^{(1)}(\lambda_n k_0 a)}. \quad (5)$$
for the cases of a soft and a hard cylindrical surface, respectively. In above relations \( \Phi^{(p)}(x) = dH^{(p)}(x)/dx \), \( p=1,2 \), denote the derivatives of Hankel functions with respect to the argument.

With the aid of extensive asymptotic analysis it can be shown that a lower bound of the number of azimuthal terms required to achieve numerical convergence of the series (4) is \( m_{cr} = \lambda_1 k_0 \min\{r, r_s\} \), which, for the isovelocity case, is given by

\[
m_{cr} = \lambda_1 k_0 \min\{r, r_s\} = \left(1 - \left(\frac{f_0}{f}\right)^2\right)^{2\pi f \min\{r, r_s\}} \frac{c_0}{c_0},
\]

where \( f_0 = c_0/4h \) denotes the cutoff frequency of the waveguide.

3. DECOMPOSITION OF THE PRESSURE FIELD

The pressure field given by the series expansion (4) is now decomposed as follows:

\[
p = p_s + p_d,
\]

where

\[
p_s = -\frac{i}{4} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} e_m H_m^{(1)}(\lambda_n k_0 r_s) g_n(z_s) J_m(\lambda_n k_0 r_c) g_n(x) \cos m\theta.
\]

Using properly Graf’s addition theorems, the component \( p_s \) takes the form

\[
p_s = \frac{i}{4} \sum_{n=1}^{\infty} g_n(z_s) g_n(x) H_m^{(1)}(\lambda_n k_0 w),
\]

where \( w = \sqrt{r_s^2 + r^2 - 2r_s r \cos \theta} \). Accordingly, \( p_s \) can be identified as the axisymmetric, unobstructed component of the acoustic pressure field in the waveguide [5],[6]. After some algebra, the following expression is derived for the diffraction component \( p_d \) of the acoustic pressure field:

\[
p_d(r, \theta, z) = -\frac{i}{4} \sum_{n=1}^{\infty} e_m H_m^{(1)}(\lambda_n k_0 r_s) g_n(z_s) \Lambda_m(\lambda_n k_0 a) H_m^{(1)}(\lambda_n k_0 r) g_n(x) \cos m\theta,
\]

where the new coefficients \( \Lambda_m(\lambda_n k_0 a) \) are given by

\[
\Lambda_m^{\text{soft}}(\lambda_n k_0 a) = \frac{J_m(\lambda_n k_0 a)}{H_m^{(1)}(\lambda_n k_0 a)}, \quad \Lambda_m^{\text{hard}}(\lambda_n k_0 a) = \frac{J_m'(\lambda_n k_0 a)}{\Phi_m^{(1)}(\lambda_n k_0 a)},
\]

for the cases of a soft and a hard cylindrical surface, respectively, and \( J_m'(x) \) stands for \( dJ_m(x)/dx \).

Accordingly, the total acoustic pressure field in the obstructed waveguide is given by

\[
p(r, \theta, z) = \frac{i}{4} \sum_{n=1}^{\infty} g_n(z_s) g_n(x) H_m^{(1)}(\lambda_n k_0 w) + p_d(r, \theta, z).
\]
On the basis of asymptotic analysis of the diffraction series (10) it can be shown[3] that a lower bound of the number of azimuthal terms required for numerical convergence is now given by $m_{\text{cr,d}} = \lambda_1k_0a$, which, for the isovelocity case, reduces to

$$m_{\text{cr,d}} = \lambda_1k_0a = \sqrt{1 - \left(\frac{f_0}{f}\right)^2\frac{2\pi f a}{c_0}}.$$  

Comparing bounds (6) and (13) we see that the decomposition (12) results in a reduction of the azimuthal terms required for numerical calculations analogous to $a/r$, for $a < r < r_s$, and to $a/r_s$, for $r_s < r < \infty$.

4. RESULTS AND DISCUSSION

The diffraction normal-mode series given by Eq. (12) is applied to the calculation of the acoustic pressure field, for an environment characterized by the following parameters: $h=250\text{m}$, $a=200\text{m}$, $z_i=2h/3$, $r_s=2000\text{m}$, $c(z)=c_0=1500\text{m/s}$. For this environment horizontal patterns of the transmission loss (in dB), at the depth of the source and for frequencies $f=500\text{Hz}$ and $f=1000\text{Hz}$, are shown in Fig. 2a-d. The case of a Dirichlet boundary condition on the surface of the scatterer (soft island) is presented in Fig. 2a,b, and the case of a Neumann boundary condition (hard island) is presented in Fig. 2c,d. As frequency increases, the edge of the shadow becomes more distinct at the rear of the island, and a trend can be observed for this boundary to become smoother. A deep

![FIG. 2. TL calculated in the case of a soft and a hard cylindrical island, for frequencies 500Hz and 1kHz.](image-url)
shadow zone appears behind the island, which is much deep in the case of a soft island. Outside the angular section defined by the tangents drawn from the source to the island, and away from the liquid-solid interface, the incident component is dominant.

The analytical solution presented herein can serve as a 3D benchmark problem in underwater acoustic modeling. Furthermore, it can easily be generalized to more realistic environments without considerable additional effort. The structure of the solution (12) for the pressure field allows independent solution of the vertical eigenvalue problem (v.e.p.). Consequently, horizontally stratified environments with general depth-dependent sound speed profiles can be handled with the aid of a robust algorithm for the calculation of the eigenvalues and eigenfunctions of the v.e.p. [7],[8]. Further extensions of the present solution to cover the corresponding problem in the presence of a cylindrical island made of an acoustic or elastic material are also possible, and they are under development.

REFERENCES


