WAVE – CURRENT – SEABED INTERACTION
OVER GENERAL BOTTOM TOPOGRAPHY

K.A. Delibassakis¹, Th.P. Gerostathi¹ & G.A. Athanassoulis²

¹ School of Technological Applications, Technological Educational
Institute of Athens, Ag. Spyridonos, Athens 12210, Greece.
² School of Naval Architecture & Marine Engineering,
National Technical University of Athens,
Heroon Polytechniou 9, Athens 15773, Greece.

ABSTRACT
A new coupled-mode model is presented for wave-current-seabed interaction, with application
to wave scattering by steady currents over general bottom topography. The vertical structure
of the scattering wave potential is represented by a series of local vertical modes containing
the propagating mode and all evanescent modes, plus an additional term accounting for the bottom
boundary condition when the bottom slope is not negligible. Using the above representation, in
conjunction with Luke’s (1967) variational principle, the wave-current-seabed interaction
problem is reduced to a coupled system of differential equations on the horizontal plane. If only
the propagating mode is retained in the vertical expansion of the wave potential, and after
simplifications, the present system is reduced to an one-equation model compatible with Kirby’s
(1984) mild-slope model (see also Liu 1990) with application to the problem of wave-current
interaction over slowly varying topography.

INTRODUCTION
The evolution of water-waves in nearshore and coastal areas, and especially, in regions where
ambient, tidal and other type of currents co-exist, is important for a variety of engineering
applications including interaction of waves with structures, coastal management, harbour
maintenance, exploitation of renewable energy sources etc. Sometimes the effects of currents on
wave transformation can be substantial, since they are responsible for Doppler shifting and for
causing wave refraction, reflection, and breaking. Also, currents could completely change the
wave energy pattern. In particular, the characteristics of surface waves present significant
variation as they propagate through non-homogeneous ambient currents, in the presence of
depth inhomogeneities in variable bathymetry regions. Thus, large amplitude waves can be
produced when obliquely propagating waves interact with opposing currents, see, e.g., Mei
(1983). This situation could be further enhanced by inshore effects due to sloping seabeds, and
has been reported to be connected with the appearance of “giant waves”, Lavrenov & Porubov
(2006). Extensive reviews on the subject of wave-current interaction in the nearshore region
have been presented by Peregrine (1976), Jonsson (1990) and Thomas & Klopman (1997).

The study of spatial evolution of water waves and the investigation of scattering of realistic
wave spectra over irregular currents, with characteristic length of variation comparable to the
dominant wavelength, and including the effects of bottom irregularities, can be supported by
theoretical models treating the simpler problem of monochromatic waves interacting with steady
inhomogeneous currents. Wave-current interaction models over slowly varying bottom
topography have been developed and studied by various authors. Under the assumption of
irrotational wave motion, Kirby (1984) derived a phase-resolving one-equation model,
generalizing the Berkhoff's mild-slope equation in regions with slowly varying depth and
ambient currents; see also Liu (1990). In the present work, a new, weakly nonlinear, coupled-mode model is developed for the wave-current-seabed interaction problem, with application to wave scattering by steady ambient currents over general bottom topography. Based on previous work by the authors (Athanasoulis & Belibassakis 1999, Belibassakis et al 2001, Gerostathis et al 2008), the vertical distribution of the scattered wave potential is represented by a series of local vertical modes containing the propagating mode and all evanescent modes, plus an additional term accounting for the bottom boundary condition when the bottom slope is not negligible (the sloping-bottom mode). Using the above representation, in conjunction with Luke's (1967) variational principle, the wave-current-seabed interaction problem is reduced to a coupled system of differential equations on the horizontal plane. If only the propagating mode is retained in the vertical expansion of the wave potential, and after simplifications and linearisation, the present system is reduced to an one-equation model (the wave-current modified mild-slope model), which after additional simplifications is shown to be compatible with Kirby's (1984) mild-slope model. The present coupled-mode system is discretized on the horizontal plane by using a second-order finite difference scheme and numerically solved by iterations. Results are presented for representative test cases corresponding to a rip current over a sloping beach in shallow water, and to a ring current in constant depth. In these cases comparisons are presented of the results obtained by the one-equation model (i.e. keeping only the propagating mode in the expansion of the wave potential) and the multi-mode system (also including the rest of the vertical modes), demonstrating the importance of the first evanescent modes and the sloping-bottom mode. The analytical structure of the present model facilitates its extension to treat fully non-linear waves, and it can be further elaborated to study wave propagation over random bottom topography and scattering by currents with general 3D structure.

THE BACKGROUND CURRENT FLOW
We consider wave propagation, in the presence of ambient, non-homogeneous current, in a variable bathymetry region; see Fig. 1. The liquid is assumed inviscid and homogeneous, and the background current flow is assumed to be steady and self-existent. Moreover, the wave flow perturbing the background current flow, which is generated by an incident wave system coming from the far up-wave region (see Fig.1), is assumed to be irrotational. A Cartesian coordinate system is used, having its origin at some point on the unperturbed free-surface ($z=0$). The $z$-axis is pointing upwards and one of the horizontal axes is taken to be (approximately) aligned
with the mean direction of the transmitted wave field; see Fig. 1. The current \( \mathbf{q} = (U_1, U_2, W) \) has been assumed to be steady and self-existent and the kinematics of this flow require that,

\[
\nabla \cdot \mathbf{q} = 0,
\]

\[
q \cdot \mathbf{n} = 0, \quad z = H(x_1, x_2), \quad q \cdot \mathbf{n} = 0, \quad z = -h(x_1, x_2),
\]

where \( \nabla \cdot \) denotes the gradient operator in 3D, and \( H \) denotes the mean set-down associated with the background current flow. Eq. (2c) is equivalently written in the form,

\[
W + U_1 \frac{\partial h}{\partial x_1} + U_2 \frac{\partial h}{\partial x_2} = 0, \quad z = -h(x_1, x_2),
\]

Assuming the current to be nearly horizontal and its velocity components to be small, the mean set-down can be neglected \( (H \approx 0) \), and the kinematical free-surface boundary condition (2b) can be linearised as follows:

\[
W = 0, \quad z = 0.
\]

The dynamics of the steady background current are described through the corresponding Bernoulli equation, stating that total energy is conserved along the streamlines (Bachelor 1967),

\[
Q = \frac{1}{2} \left| \mathbf{q}_a \right|^2 + \frac{P}{\rho} + g z = \text{const} = \frac{1}{2} \left| \mathbf{q}_a \right|^2 + \frac{P_a}{\rho} + g z_a,
\]

where \( \rho \) is the liquid density, \( g \) is the gravitational acceleration and \( z_a \) denotes the vertical position of each streamline at infinity. Taking \( P_a \) to be the static pressure of the fluid at rest, obtained by the superposition of the atmospheric and the hydrostatic pressure at infinity,

\[
P_a = P_a - \rho g z_a,
\]

we finally obtain the following equation

\[
\frac{P}{\rho} = \frac{1}{2} \left( \left| \mathbf{q}_a \right|^2 - \left| \mathbf{q} \right|^2 \right) + \frac{P_a}{\rho} - g z.
\]

The total (current and wave) flow velocity \( \mathbf{u} \), as well as the background current flow velocity \( \mathbf{q} \), both satisfy Euler equations. Subtracting these equations by parts and omitting approximately the terms associated with the vorticity of the background flow \( (\nabla \times \mathbf{q}) \), which is assumed to be weak, we finally obtain the following approximate Bernoulli equation,

\[
\frac{\rho}{\rho} + \frac{\partial \varphi}{\partial t} + \mathbf{q} \cdot \nabla \varphi + \frac{1}{2} \left| \nabla \varphi \right|^2 = 0,
\]

where \( \varphi(x_1, x_2, z, t) \) denotes the (disturbance) wave potential, \( \nabla \varphi = \mathbf{u} - \mathbf{q} \), and \( p \) is the wave-induced pressure, defined as the difference between the total \( (P) \) and the background \( (P) \) pressure, \( p = P - P_a \). Using the above equations, we finally obtain the following Bernoulli equation expressing the conservation of energy of the studied wave-current problem:

\[
P - P_a = -\rho \left[ g z + \frac{\partial \varphi}{\partial t} + \frac{1}{2} \left( \left| \mathbf{u} \right|^2 - \left| \mathbf{q}_a \right|^2 \right) \right].
\]

**VARIATIONAL FORMULATION**

The variational principle governing the total fluid motion is formulated using Luke’s (1967) functional, that is based on integration of pressure in the domain (see also Massel 1989).
\[ \mathcal{L} = \iiint_{\xi \in \xi, \eta \in \eta, \tau \in \tau} \iint_{z = \phi(x_1, x_2, \tau)} \left( P_u - P_g \right) dz \, dx_1 dx_1 dt, \]  

where \( \phi(x_1, x_2, \tau) \) denotes the free-surface elevation associated with the total wave and current flow. Using (7), the above equation is written as follows

\[ \mathcal{L} = -\rho \iiint_{\xi \in \xi, \eta \in \eta, \tau \in \tau} \iint_{z = \phi(x_1, x_2, \tau)} \left( g \frac{\partial \phi}{\partial t} + \frac{1}{2} \left( |u|^2 - |q_e|^2 \right) \right) dz \, dx_1 dx_1 dt. \]  

Using the fact that the current has been assumed essentially horizontal \((W \ll U_1, U_2)\) and slowly varying \((|q|^2 \approx |q_e|^2)\), the quantity \(|u|^2 - |q_e|^2\) in the right-hand side of Eq.(9) can be approximated as follows: \(|u|^2 - |q_e|^2 \approx U_1 \frac{\partial \phi}{\partial x_1} + U_2 \frac{\partial \phi}{\partial x_2} + (\nabla \phi)^2\). Using the latter and requiring the above functional to be stationary, \(\partial \mathcal{L} = 0\), and after carrying out the algebra, we finally obtain the following variational equation (see also Massel 1989, Eq.1-35)

\[ -\rho \iiint_{\xi \in \xi, \eta \in \eta, \tau \in \tau} \left[ \frac{1}{2} \left( U_1 \frac{\partial \phi}{\partial x_1} + U_2 \frac{\partial \phi}{\partial x_2} + (\nabla \phi)^2 \right) + \frac{\partial \phi}{\partial t} + g \eta \right] \delta \eta + \left[ \frac{\partial \eta}{\partial t} + U_1 \frac{\partial \phi}{\partial x_1} \frac{\partial \eta}{\partial x_1} + U_2 \frac{\partial \phi}{\partial x_2} \frac{\partial \eta}{\partial x_2} - \left( W + \frac{\partial \phi}{\partial x} \right) \delta \phi + \left[ \frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial y} \delta x + \frac{\partial \phi}{\partial \xi} \delta \xi \right] \delta \phi \right] \delta x_1 dx_1 dt = 0, \]  

where also the continuity equation concerning the background current, Eq.(1), and the bottom boundary condition, Eq. (2c), have been used. The equations governing the fully non-linear wave-current problem are derived from the above principle, Eq. (10).

**THE NONLINEAR COUPLED-MODE SYSTEM**

Following previous works (e.g., Athanassouls & Belibassakis 2002, Belibassakis & Athanassouls 2006), we use the following local-mode series expansion of the wave potential

\[ \phi(x, z) = \varphi_0(x, z) + \varphi_1(x, z) + \ldots + \varphi_n(x, z), \]

in the variable bathymetry region, where \( x = (x_1, x_2) \). In the above expansion, the mode \( n=0 \) (i.e., \( \varphi_0(x, z) \)) denotes the *propagating mode* and the remaining terms, \( n=1,2, \ldots \), are the *evanescent modes*. The term \( \varphi_1(x, z) \) is a correction term, called the *sloping-bottom mode*, properly accounting for the satisfaction of the Neumann bottom boundary condition on the non-horizontal parts of the bottom. The term \( \varphi_2(x, z) \) is a corresponding term introduced to ensure the satisfaction of the free-surface boundary condition. The rest of functions \( \varphi_n(x, z), \) \( n=0,1,2, \ldots \), appearing in Eq. (11), are obtained as the eigenfunctions of the following local vertical Sturm-Liouville problems formulated with respect to the local depth and the intrinsic frequency \( \sigma = \omega - \mathbf{U} \cdot \mathbf{k} \):
\[ Z^*_{n} + k_n^2 Z_n = 0 \ , -h(x) < z < \eta(x,t), \]  
\[ Z^*_{n}(-h(x)) = 0 \ , \ z = -h(x) \ , \ Z_n(0) = 0 \ , \ z = \eta(x,t), \]  
where the prime denotes differentiation with respect to \( z \). The parameter \( \mu \) is taken to be the intrinsic frequency parameter \( \mu = \sqrt{\omega^2 / k} \), where \( \omega \) denotes the characteristic absolute wave frequency. The definition of the intrinsic frequency depends also on the vector wavenumber \( k \), a quantity which is associated with the wave kinematics (see e.g. Jonsson 1990), and thus, it is clearly dependent on the solution \( \varphi(x_1, x_2; z) \). This fact introduces intrinsic nonlinearity to the wave-current scattering problem, and iterations are necessary for its solution. The local vertical eigenfunctions are obtained from Eqs. (12, 13) as follows:
\[ Z_0 = \frac{\cosh[k_n(z + h)]}{\cosh[k_n(h + h)]} \ , \ Z_n = \frac{\cos[k_n(z + h)]}{\cos[k_n(h + h)]}, \quad n = 1, 2, \ldots, \]
where the wavenumbers \( k_n = k_n(h, \eta) \) are obtained as a solution to the local dispersion relation,
\[ \sigma^2 = k_n g \tan[k_n(h + h)] = -k_n g \tan[k_n(h + h)], \quad n \geq 1. \]
As concerns the sloping-bottom mode \( \varphi_{z=1} \), a specific convenient form of the function \( Z_{z=1}(z; x) \) is given by
\[ Z_{z=1} = \frac{\mu h_0 - 1}{2h_0(h + h)} (z + h)^2 + \frac{1}{h_0} (z + h) + \frac{1}{2} \frac{(\mu h_0 + 1)}{h_0}, \]
where the parameters \( \mu_0, h_0 > 0 \) are positive constants, having the following properties:
\( Z_{z=1}'(z = \eta) - \mu_0 Z_{z=1}(z = \eta) = 0 \ , \ Z_{z=1}'(z = -h) = 1/\mu_0 \). However, other forms are also valid. Finally, as concerns the free surface mode \( \varphi_{z=2} \), a specific convenient form of the function \( Z_{z=2}(z; x) \) is given by,
\[ Z_{z=2}(z; h, \eta) = \frac{\mu h_0 - 1}{2(h + h)} (z + h)^2 - \frac{1}{2} \frac{1}{h_0} (z + h), \]
having the properties: \( Z_{z=2}'(z = \eta) - \mu_0 Z_{z=2}(z = \eta) = 1/\mu_0 \ , \ Z_{z=2}'(z = -h) = 0 \), however other forms are also possible; see also Belibassakis & Athanassoulis (2006) for more details. Using the representation (11) in the variational equation (10), and after the algebra, we obtain an equivalent reformulation of the wave-current scattering problem in the form of the following nonlinear Coupled Mode System (CMS) of equations,
\[ \sum_{n=2}^{n} \left( A_{mn}(\eta) \nabla^2 \varphi_n + B_{mn}(\eta) \cdot \nabla \varphi_n + C_{mn}(\eta) \varphi_n \right) + + \frac{\partial \eta}{\partial t} + U \cdot \nabla \eta - W = 0, \]  
\[ \sum_{n=2}^{n} \left( \frac{\partial \varphi_n}{\partial t} + \left[ [Q_n] \right]_{z=0} \frac{\partial \eta}{\partial t} + UV \varphi_n + W \varphi_n \frac{\partial Z_{z=1}}{\partial z} \right) + g \eta + \frac{1}{2} \left( U_{i}^2 + U_{j}^2 + W^2 \right) + \]
\[ - \sum_{n=2}^{n} \sum_{m=2}^{m} \left( a^{(01)}_{mn}(\eta) \varphi_n \frac{\partial \varphi_n}{\partial x} + a^{(11)}_{mn}(\eta) \frac{\partial \varphi_n}{\partial x} \frac{\partial \varphi_n}{\partial x} + b_{mn}(\eta) \frac{\partial \varphi_n}{\partial x} + c_{mn}(\eta) \varphi_n \right) = 0, \]
\[ A_{m} (\eta) = \int_{z=-h(x)}^{z=h(x)} Z_n(z,h,\eta) Z_m(z,h,\eta) \, dz, \quad (19a) \]

\[ B_{m} (\eta) = 2 (\nabla Z_n \cdot \nabla Z_m) + \left[ \left( \nabla Z_n \cdot \nabla h + \frac{\partial Z_n}{\partial z} \right) Z_m \right]_{z=-h(x)}^{z=h(x)} + \nabla \eta [Z_n Z_m]_{z=-h(x)}^{z=h(x)}, \quad (19b) \]

\[ C_{m} (\eta) = \left[ \frac{\partial^2 Z_n}{\partial z^2}, Z_m \right]_{z=-h(x)}^{z=h(x)} + \left[ \nabla Z_n \cdot \nabla h + \frac{\partial Z_n}{\partial z} \right] Z_m \right]_{z=-h(x)}^{z=h(x)} + \left[ \nabla Z_n \cdot \nabla \eta - \frac{\partial Z_n}{\partial z} \right] Z_m \right]_{z=-h(x)}^{z=h(x)}, \quad (19c) \]

where the brackets denote the inner product in the vertical interval:

\[ \langle f(z), g(z) \rangle = \int_{z=-h(x)}^{z=h(x)} f(z) g(z) \, dz. \]

Moreover, the matrix coefficients \( a_{m}^{(0)}, a_{m}^{(1)}, b_{m}, \) and \( c_{m} \) are similarly defined; see also Athanassoulis & Belibassakis (2007).

**The weakly nonlinear CMS**

By assuming small wave amplitudes and considering the wave velocity field to be time harmonic with angular frequency \( \omega \), we introduce the complex velocity potential

\[ \phi(x_1,x_2,z,t) = \Re \left\{ \phi(x_1,x_2,z) \exp(-i\omega t) \right\}. \quad (20) \]

In this case, the variational principle (10) simplifies as follows

\[ \rho \int_{z_1}^{z_2} \int_{x_1}^{x_2} \left[ \frac{1}{g} \left( -i\omega U \cdot \nabla \right)^2 \phi + \frac{\partial \phi}{\partial z} \right] \nabla U \cdot (-i\omega U \cdot \nabla) \phi \right]_{z=0}^{z=h(x)} + \left[ \frac{\partial \phi}{\partial z} + \nabla \phi \nabla h \right]_{z=-h(x)}^{z=0} \left( \nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} \right) dz \right] \delta \phi \right)_{x_1}^{x_2} = 0, \quad (21) \]

where, from now on, \( \phi = \phi(x_1,x_2,z) \) denotes the complex wave potential. Furthermore, noting that the term \( (-i\omega U \cdot \nabla)^2 \phi + (\nabla U) \cdot (-i\omega U \cdot \nabla) \phi \), appearing in the integral on the mean free-surface \( z=0 \), can also be written as follows:

\[ A = -\omega^2 \phi - 2i\omega (U \cdot \nabla) \phi - \omega (U \cdot \nabla) \phi + \nabla \cdot (U (U \cdot \nabla) \phi), \]

the variational principle for small-amplitude time-harmonic wave motion, Eq. (21), is finally put in the form

\[ \int_{z_1}^{z_2} \int_{x_1}^{x_2} \left[ \int_{z=-h(x)}^{z=0} \left( \nabla^2 \phi + \frac{\partial^2 \phi}{\partial z^2} \right) dz + \left[ \frac{\partial \phi}{\partial z} + \nabla \phi \nabla h \right]_{z=-h(x)}^{z=0} \right] \delta \phi \right)_{x_1}^{x_2} = 0. \quad (22) \]

Also, in this case, the local-mode series expansion of the wave potential is simplified as follows

\[ \phi(x,z) = \sum_{n=1}^{\infty} \phi_n(x) \cdot Z_n(z;x), \quad (23) \]

where the functions \( Z_n(z;x), n = 0,1,2, \ldots \), appearing in Eq. (23), are obtained as the eigenfunctions of local vertical Sturm-Liouville problems (12, 13), formulated with respect to the local depth and the local intrinsic frequency \( \sigma = \omega - U \cdot k \) in the vertical interval \(-h(x) < z < 0\). Thus, in the examined case the local vertical eigenfunctions are given by
\[ Z_0 = \frac{\cosh[k_z(z + h)]}{\cosh(k_z h)}, \quad Z_n = \frac{\cos[k_z(z + h)]}{\cos(k_z h)}, \quad n = 1, 2, \ldots, \quad (24) \]

where the wavenumbers \( k_z \) are obtained as a solution to the (simplified) local dispersion relation associated with the intrinsic frequency:

\[ \sigma^2 = k_z g \tanh(k h) = -k_z g \tan(k h). \quad (25) \]

As concerns the sloping-bottom mode \( \varphi_{z1} Z_{z1} \), a specific alternative form of the function \( Z_{z1}(z, x) \) used in this case is defined by

\[ Z_{z1} = h \left( \frac{z}{h} + \frac{(z/h)^2}{2} \right), \quad (26) \]

having the following properties: \( Z'_{z1}(z = -h) = 1 \), \( Z_{z1}(z = -h) = 0 \), and \( Z'_{z1}(z = 0) = Z_{z1}(z = 0) = 0 \) (see also the discussion by Athanassoulis & Bellbassakis 1999).

Using the representation (23), in conjunction with the properties of \( Z_n \), we obtain

\[ \frac{\partial \varphi}{\partial z} \bigg|_{z=0} = \mu \varphi = \frac{\sigma^2}{g} \varphi, \quad \text{on} \quad z = 0. \quad (27) \]

Introducing the above result in the last form of the variational principle, Eq. (22), we obtain

\[ \int_{t_1}^{t_2} \int_{x_1}^{x_2} \left[ \int_{z=-h(z, x)}^{z=0} \left( \nabla^2 \varphi + \frac{\partial^2 \varphi}{\partial z^2} \right) dz + \left[ \frac{\partial \varphi}{\partial z} + \nabla \varphi \nabla h \right] \bigg|_{z=-h(z, x)} - \left[ \frac{A}{g} + \frac{\sigma^2 \varphi}{g} \right] \bigg|_{z=0} \right] \delta \varphi = 0. \quad (28) \]

Using the local-mode representation (23) in the variational equation (28), we finally obtain the weakly nonlinear coupled-mode system (CMS) of equations, in the form

\[ \sum_n \left[ a_{mn} \nabla^2 \varphi_m + \left[ b_{mn} + \frac{2i\omega}{g} \right] \nabla \varphi_n + \left[ c_{mn} + \frac{\omega^2 - \sigma^2}{g} + \frac{i\omega}{g} \left( \nabla \cdot U \right) \right] \varphi_n - \frac{1}{g} \nabla \cdot \left( \frac{U}{\left( \nabla \cdot V \right) \varphi_n} \right) \right] = 0, \]

\[ m = -1, 0, 1, \ldots, \quad (29) \]

In Eqs. (29) the coefficients \( a_{mn}, b_{mn}, c_{mn} \) are defined by:

\[ a_{mn} = \left\langle Z_m, Z_n \right\rangle, \quad b_{mn} = 2 \left\langle \nabla Z_m, Z_n \right\rangle + Z_n (-h) Z_m (-h) \nabla h, \quad \text{and} \]

\[ c_{mn} = \left\langle \nabla^2 Z_m + \frac{\partial^2 Z_m}{\partial z^2}, Z_n \right\rangle + \left. \left( \frac{\partial Z_m}{\partial z} \right) \bigg|_{z=-h} \right. + \nabla Z_m \nabla h \bigg|_{z=-h} Z_n (-h). \quad \text{(30a, b, c)} \]

**The one-equation model**

In the case of slowly varying bottom and current(s), a significant simplification of the above CMS (29) can be obtained by keeping only the propagating mode \((n = 0)\) in the local-mode series expansion of the wave potential (23), which essentially describes the propagation features. In this case, the above coupled-mode system is reduced to the following *one-equation model* on the horizontal plane,

\[ a_{00} \nabla^2 \varphi_0 + \left[ b_{00} + \frac{2i\omega}{g} \right] \nabla \varphi_0 - \frac{1}{g} \nabla \cdot \left( \frac{U}{\left( \nabla \cdot V \right) \varphi_0} \right) + \left[ c_{00} + \frac{\omega^2 - \sigma^2}{g} + \frac{i\omega}{g} \left( \nabla \cdot U \right) \varphi_0 \right] = 0. \quad (31) \]

The coefficients \( a_{00}, b_{00}, c_{00} \) are given by Eqs. (30) for \( m = n = 0 \), and are calculated to be:
\[ a_{00} = \int_{z=\eta}^{0} Z_0^2 dz = \frac{1}{2k_0} \tanh\left( k_0 \eta \right) \left( 1 + \frac{2k_0 \eta}{\sinh(2k_0 \eta)} \right) = \frac{1}{g} CC_\eta, \tag{32a} \]
\[ b_{00} = \nabla a_{00} = \frac{1}{g} \nabla CC_\eta = 2 \left( \nabla Z_0, Z_0 \right) + Z_0^2 (-h) \nabla h, \tag{32b} \]
\[ c_{00} = k_0^2 a_{00} + \left( \nabla^2 Z_0, Z_0 \right) + \nabla Z_0 \nabla h Z (-h) = k_0^2 a_{00} + c_{00}^{(2)}. \tag{32c} \]

We note here that the coefficient \( c_{00}^{(2)} = \left( \nabla^2 Z_0, Z_0 \right) + \nabla Z_0 \nabla h Z (-h) \) contains extra terms proportional to first and second horizontal derivatives of the depth function (proportional to bottom slope and curvature), as well as first and second horizontal derivatives of the horizontal current velocity components \( U_1 \) and \( U_2 \). Using the above expressions of the coefficients in the one-equation model (31) and multiplying by \( g \), it is easily seen that the latter takes the form
\[
\nabla \left( CC_\eta \nabla \varphi_0 \right) - \nabla \cdot \left[ \mathbf{U} \left[ \left( \mathbf{U} \cdot \nabla \right) \varphi_0 \right] \right] + 2i\omega \mathbf{U} \cdot \nabla \varphi_0 + \left[ k_0^2 CC_\eta + c_{00}^{(2)} \right] + \omega^2 - \sigma^2 + i\omega \left( \nabla \cdot \mathbf{U} \right) \varphi_0 = 0, \tag{33} \]

which will be called the modified mild-slope equation for wave scattering by ambient current in general bottom topography. We note here that the model (33), in the case of no current \( (U=0) \), exactly reduces to the modified mild slope equation derived by Massel (1993) and Chamberlain & Porter (1995). Furthermore, it is worth noticing here that if the term \( c_{00}^{(2)} \) is omitted, then Eq. (33) reduces exactly to the mild slope equation derived by Kirby (1984), in its time-harmonic form; see, e.g., Chen et al (2005).

**Numerical Results and Discussion**

In this section, numerical results are presented for two representative test cases corresponding to a rip current over a sloping beach in shallow water, and to a ring current in constant depth.

**The case of rip current**

Wave-induced rip currents, created by longshore currents converging into periodic rips and forming independent coastal circulation cells, play an important role in coastal morphodynamics. Also, rip currents, in conjunction with local amplification of wave energy, are responsible for many accidents in beaches. In order to illustrate the effects of a rip current, in conjunction with slow changes of the bathymetry, on the wave scattering, we examine a sloping beach of uniform slope 1/50. Using the similar expressions introduced by Chen et al (2005), the structure of the rip current \( \mathbf{U} = (U_1, U_2) \) considered in this example is modelled as follows:

\[ U_1 = -0.0681 x_1 \frac{P \left( x_1 / 76.2 \right) F \left( x_1 / 76.2 \right)}{\int_{x_1 = 76.2}^{\infty} F \left( \tau \right) d\tau}, \tag{34a} \]
\[ U_2 = -0.256 \left[ 2 - \left( x_1 / 76.2 \right)^2 \right] \frac{F \left( x_1 / 76.2 \right)}{\int_{0}^{\infty} F \left( \tau \right) d\tau}, \tag{34b} \]

where \( F \left( \tau \right) = (2\pi)^{-1/2} \exp \left( -\tau^2 / 2 \right) \).

The structure of the current is shown in Figs. 2 and 3, and, its maximum value is selected to be 0.5 m/s. We consider unit-amplitude harmonic waves of period \( T=14s \) (corresponding to a swell) propagating along the \( x_1 \)-axis (normally to the bottom contours). In this case, the phase velocity of the incident wave is \( C=5.5 \) m/s. The scattered wave field by the above configuration as calculated by means of the solution of the present modified mild-slope equation (using only one mode) is plotted in Fig.2a (phase) and Fig.3a (amplitude). The same result calculated using three
modes \( (n=-1,0,1) \) is comparatively plotted in Fig.2b (phase) and Fig.3b (amplitude). In the examined case, a 350m by 200m horizontal domain is considered, discretised by using 151 equidistant points along each horizontal direction. In this case, a significant focusing of wave energy as the shallow-end of the variable bathymetry region is approached is observed, taking place immediately shorewards the formation of the rip current. Also, in this area (where energy is maximized) the greatest differences between the one and three mode solution are observed. To illustrate the role and the relative significance of the various modes, the amplitudes of the propagating mode \( \varphi_i(x,y) \), the first evanescent mode \( \varphi_1(x,y) \) and the sloping-bottom mode \( \varphi_{-1}(x,y) \), are plotted in Fig.4 a), b) and c), respectively. We observe in these figures that the main effects of wave refraction and diffraction are carried out by the propagating mode. This fact justifies the use of the simplified, one-equation MMS model to obtain approximate results. However, as shown by Athanassoulis et al (2003), the applicability of the latter model is restricted to slowly varying bottom topographies. Moreover, in Fig.4 we observe that the first evanescent mode \( \varphi_1(x,y) \) is one order of magnitude less than the propagating mode. This mode is clearly connected with the non-horizontal part of the bottom. Furthermore, the sloping-bottom mode \( \varphi_{-1}(x,y) \) is of the same order of magnitude, in the swallow area of the bottom.

![Image of scattered wave field](image1)

**Figure 2** - Real part of the scattered wave field on the free surface by a rip current in a sloping (1:50) beach region: (a) 1 mode solution, (b) 3 mode solution.

![Image of scattered wave field](image2)

**Figure 3** - Amplitude of the scattered wave field by a rip current in a sloping (1:50) beach region, with depths ranging from 4m to 0.5m: (a) 1 mode solution, (b) 3 mode solution.

**The case of a ring current**

In this example we consider wave scattering by a ring current in constant depth \( h=10 \)m, examined also by Chen et al (2005). Such current structure is commonly seen in open sea and coastal areas, and has important impact on physical and biological processes; see Mapp et al (1989). Following Yoon & Liu (1989), the background current flow associated with the vortex.
Figure 4 - Modal amplitudes of the scattered wave field by a rip current in a sloping beach region: (a) the propagating mode \( (n=0) \), (b) the first evanescent mode \( (n=1) \) and (c) the sloping bottom mode \( (n=-1) \).

Figure 5 - Real part of the scattered wave field on the free surface: (a) 1 mode solution and (b) 3 modes solution.

Figure 6 - Comparison of modal amplitudes: (a) the propagating mode \( (n=0) \), and (b) the first evanescent mode \( (n=1) \).

ing is defined by,

\[
U_r = 0, \quad U_\theta = \begin{cases} 
C_5 \left( r / R_i \right)^N, & r \leq R_i \\
C_5 \exp \left[ -(R_2 - r)^2 / R_2^2 \right], & r \geq R_i 
\end{cases}
\]  
(35)
where \(U_r\) and \(U_\theta\) denote the radial and tangential components of the horizontal flow \(U\), in a cylindrical-polar coordinate system \((R=|x|, \theta = \tan^{-1}(x_2/x_1))\) with origin at the center of the vortex ring. The following values of the parameters (see Mapp et al. 1985) have been selected to describe the vortex ring:

\[
C_s = 0.9 \text{ m/sec}, \quad C_v = 1.0 \text{ m/sec}, \quad N = 2,
\]

\[
R_1 = 343.706 \text{ m}, \quad R_2 = 384.881 \text{ m}, \quad R_3 = 126.830 \text{ m}.
\]

The vortex ring creates a shearing current with maximum tangential velocity 1 m/s, see Fig. 5. We consider unit-amplitude harmonic waves of period \(T=20\) s, propagating along the \(x_1\)-axis, with phase velocity \(C=9.74\) m/s, and scattered by the above vortex-ring current. In the examined case, a 2 km by 2 km horizontal domain is considered, which is discretised by using 201 equidistant points along each horizontal direction.

Numerical results obtained by the present model using one and three modes are shown in Fig. 5 as concerns the real part of the wave field on the free surface. The focusing and defocusing of wave energy in the area downwave the vortex ring are well reproduced. In this case, the differences between the two solutions are very small, due to the fact that the bottom is flat, and thus, the sloping bottom mode is zero (by definition). This fact is clearly illustrated in Fig. 6, where the amplitude of the propagating (Fig. 6a) and the first evanescent mode (Fig. 6b) are comparatively plotted. It is seen in this example that the first evanescent mode is one order of magnitude less than the propagating mode and thus, the one-mode solution obtained using the one-equation model Eq. (31) is an excellent approximation.

CONCLUDING REMARKS

A coupled-mode model for the wave-current-seabed interaction problem is presented, with application to wave scattering by ambient currents in variable bathymetry regions. Based on an appropriate variational principle, in conjunction with a rapidly-convergent local-mode series expansion of the wave potential in a finite subregion containing the current variation and the bottom irregularity, the present system can be considered as a generalization of the one derived by Athanassoulis & Belibassakis 2002 for the propagation of waves in variable bathymetry regions. The key feature of the present method is the introduction of an additional mode, describing the influence of the bottom slope, and accelerating the convergence of the local-mode series. If only the propagating mode is retained in the vertical expansion of the wave potential, the above coupled-mode system is reduced to the one-equation enhanced model called the modified mild-slope equation for wave-current flow, generalizing the corresponding mild-slope equation derived by Kirby (1984). Finally, the analytical structure of the present system facilitates its extension to various directions as, e.g., to non-linear wave-current scattering problems and more general current profiles.

ACKNOWLEDGMENTS

The present work has been supported by the Operational Program for Educational and Vocational Training II (EPEAEK II) and particularly the PYTHAGORAS II project. The project is co-funded by the European Social Fund (75%) and National Resources (25%).

REFERENCES


KEYWORDS

Water waves in variable bathymetry, wave-current-seabed interaction.